Capacitance and Dielectrics

Definition of Capacitance

Consider two conductors carrying charges of equal magnitude and opposite sign, as shown in Figure 1. Such a combination of two conductors is called a capacitor.The conductors are called *plates.* A potential difference ΔV exists between the conductors due to the presence of the charges.

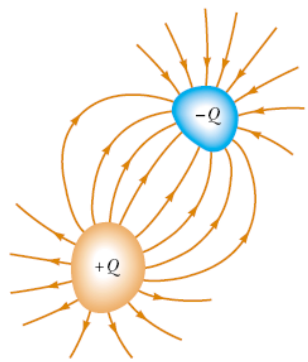


Figure (1)

The capacitance **C** of a capacitor is defined as the ratio of the magnitude of the charge on either conductor to the magnitude of the potential difference between the conductors:

$$C \equiv \frac{Q}{\Delta V}$$

From this Equation we see that capacitance has SI units of coulombs per volt. The SI unit of capacitance is the farad (F), which was named in honor of Michael Faraday:

1 F = 1 C/V

Calculating Capacitance

We can derive an expression for the capacitance of a pair of oppositely charged conductors in the following manner: assume a charge of magnitude Q, and calculate the potential difference using the techniques described in the preceding chapter. We then use the expression $C = Q/\Delta V$ to evaluate the capacitance.

The electric potential of the sphere of radius R is simply $k_e\, Q/R$, and setting $V = 0$ for the infinitely large shell, we have

$$C = \frac{Q}{\Delta V} = \frac{Q}{k_e Q/R} = \frac{R}{k_e} = 4\pi\epsilon_0 R$$

This expression shows that the capacitance of an isolated charged sphere is proportional to its radius and is independent of both the charge on the sphere and the potential difference.

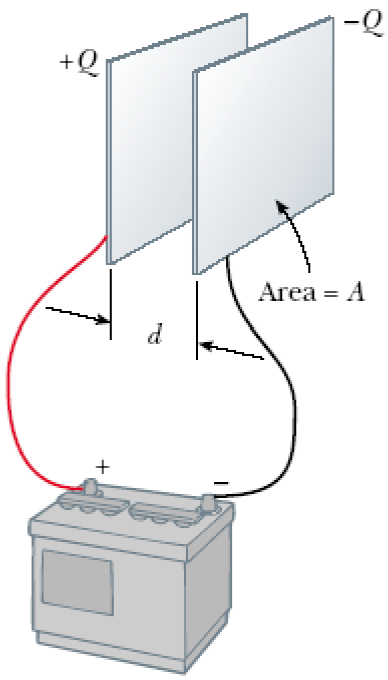


Figure (2)

Parallel-Plate Capacitors

Two parallel metallic plates of equal area A are separated by a distance d, as shown in Figure 2. One plate carries a charge Q , and the other carries a charge $-Q$. The surface charge density on either plate is σ = Q/A. If the plates are very close together (in comparison with their length and width), we can assume that the electric field is uniform between the plates and is zero elsewhere. The value of the electric field between the plates is

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}$$

Because the field between the plates is uniform, the magnitude of the potential difference between the plates equals Ed ; therefore,

$$\Delta V = Ed = \frac{Qd}{\epsilon_0 A}$$

$$C = \frac{Q}{\Delta V} = \frac{Q}{Qd/\epsilon_0 A}$$

$$C = \frac{\epsilon_0 A}{d}$$

That is, the capacitance of a parallel-plate capacitor is proportional to the area of its plates and inversely proportional to the plate separation.

Example 1 Parallel-Plate Capacitor

A parallel-plate capacitor with air between the plates has an area $A = 2.00 \times 10^{-4}$ m^2 and a plate separation $d = 1.00$ mm. Find its capacitance.

Solution

$$C = \frac{\epsilon_0 A}{d} = \frac{(8.85 \times 10^{-12}\ \mathrm{C^2/N \cdot m^2})(2.00 \times 10^{-4}\ \mathrm{m^2})}{1.00 \times 10^{-3}\ \mathrm{m}}$$

$$= 1.77 \times 10^{-12}\ \mathrm{F} = \boxed{1.77\ \mathrm{pF}}$$

Combinations of Capacitors

(1) Parallel Combination

Two capacitors connected as shown in **Figure 5a** are known as a ***parallel combination*** of capacitors. The individual potential differences across capacitors connected in parallel are the same and are equal to the potential difference applied across the combination.

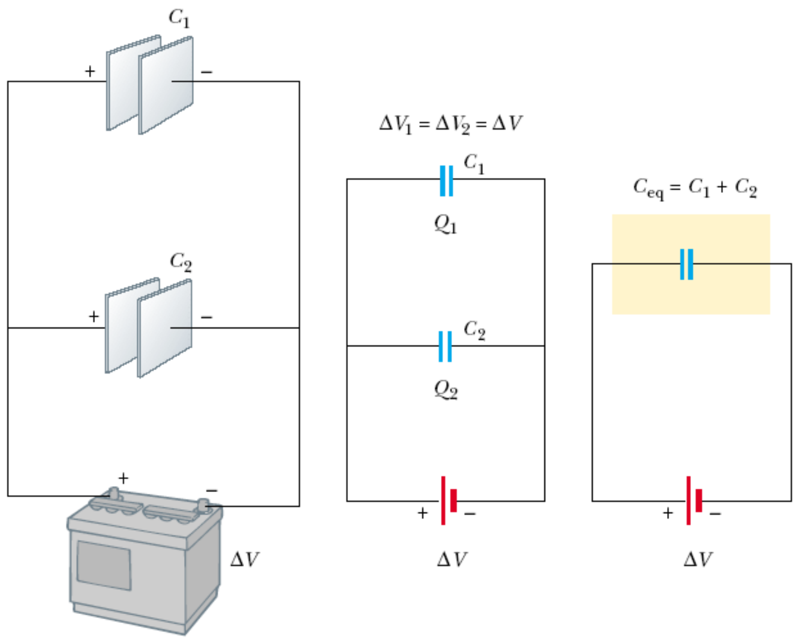


Figure (5)

Let us call the maximum charges on the two capacitors Q_1 and Q_2. The *total charge* Q stored by the two capacitors is

$$Q = Q_1 + Q_2$$

That is, the total charge on capacitors connected in parallel is the sum of the charges on the individual capacitors. Because the voltages across the capacitors are the same, the charges that they carry are

$$Q_1 = C_1 \Delta V \qquad Q_2 = C_2 \Delta V$$

Suppose that we wish to replace these two capacitors by one ***equivalent capacitor*** having a capacitance C_{eq}, as in Figure 4c.

$$Q = C_{eq}\,\Delta V$$

$$C_{eq}\,\Delta V = C_1\,\Delta V + C_2\,\Delta V$$

$$C_{eq} = C_1 + C_2 \qquad \text{(parallel combination)}$$

If we extend this treatment to three or more capacitors connected in parallel, we find the equivalent capacitance to be

$$C_{eq} = C_1 + C_2 + C_3 + \cdots \qquad \text{(parallel combination)}$$

Thus, the equivalent capacitance of a parallel combination of capacitors is the algebraic sum of the individual capacitances and is greater than any of the individual capacitances.

(2) Series Combination

Two capacitors connected as shown in Figure 6a and the equivalent circuit diagram in Figure 6b are known as a ***series combination*** of capacitors.

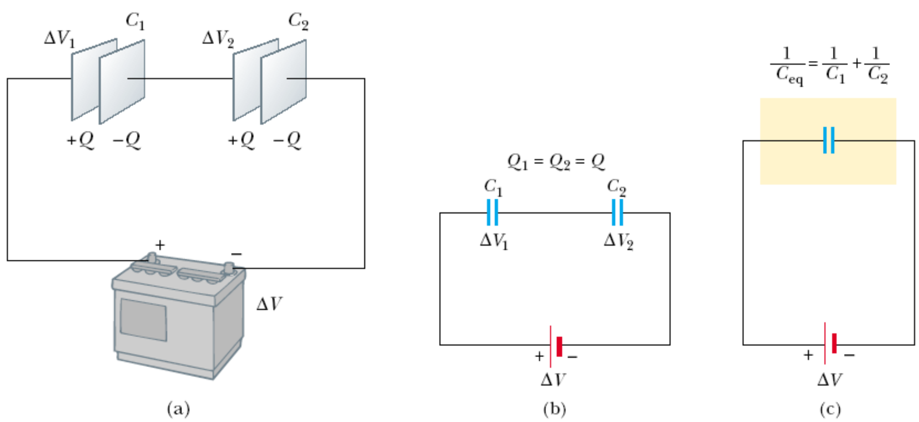


Figure (6)

The charges on capacitors connected in series are the same. From Figure 6a, we see that the voltage $\boldsymbol{\Delta V}$ across the battery terminals is split between the two capacitors:

$$\Delta V = \Delta V_1 + \Delta V_2$$

where $\boldsymbol{\Delta V_1}$ and $\boldsymbol{\Delta V_2}$ are the potential differences across capacitors $\boldsymbol{C_1}$ and $\boldsymbol{C_2}$, respectively. In general, **<u>the total potential difference across any number of capacitors connected in series is the sum of the potential differences across the individual capacitors</u>**.

$$\Delta V = \frac{Q}{C_{eq}}$$

Because we can apply the expression $Q = C\,\Delta V$ to each capacitor shown in Figure 5b, the potential differences across them are

$$\Delta V_1 = \frac{Q}{C_1} \qquad \Delta V_2 = \frac{Q}{C_2}$$

$$\frac{Q}{C_{eq}} = \frac{Q}{C_1} + \frac{Q}{C_2}$$

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} \qquad \text{(series combination)}$$

When this analysis is applied to three or more capacitors connected in series, the relationship for the equivalent capacitance is

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots \qquad \text{(series combination)}$$

This shows that **<u>the inverse of the equivalent capacitance is the algebraic sum of the inverses of the individual capacitances and the equivalent capacitance of a series combination is always less than any individual capacitance in the combination</u>**.

Example Equivalent Capacitance

Find the equivalent capacitance between *a* and *b* for the combination of capacitors shown in Figure 7a. All capacitances are in microfarads.

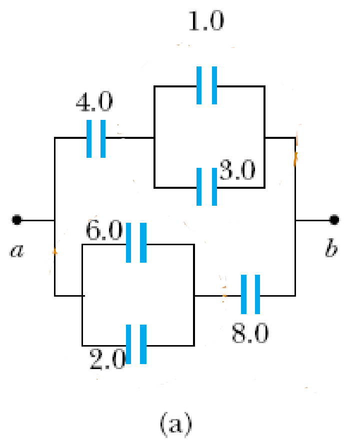


Figure (7)

Energy Stored in a Charged Capacitor

Suppose that $\boldsymbol{q}$ is the charge on the capacitor at some instant during the charging process. At the same instant, the potential difference across the capacitor is $\Delta V = q/C$. We know that the work necessary to transfer an increment of charge $\boldsymbol{dq}$ from the plate carrying charge $\boldsymbol{-q}$ to the plate carrying charge $\boldsymbol{q}$ (which is at the higher electric potential) is

$$dW = \Delta V \, dq = \frac{q}{C} \, dq$$

The total work required to charge the capacitor from $q = 0$ to some final charge $q = Q$ is

$$W = \int_0^Q \frac{q}{C} \, dq = \frac{1}{C} \int_0^Q q \, dq = \frac{Q^2}{2C}$$

The work done in charging the capacitor appears as electric potential energy ***U*** stored in the capacitor. We can express the potential energy stored in a charged capacitor in the following forms:

$$U = \frac{Q^2}{2C} = \tfrac{1}{2} Q \, \Delta V = \tfrac{1}{2} C (\Delta V)^2$$

$$\Delta V = Ed.$$

$$C = \epsilon_0 A/d$$

$$U = \tfrac{1}{2} \frac{\epsilon_0 A}{d} (E^2 d^2) = \tfrac{1}{2} (\epsilon_0 A d) E^2$$

Because the volume occupied by the electric field is ***Ad***, the *energy per unit volume* $u_E = U/Ad$, known as the *energy density*, is

$$u_E = \tfrac{1}{2} \epsilon_0 E^2$$

The energy density in any electric field is proportional to the square of the magnitude of the electric field at a given point

References

This lecture is a part of chapter 26 from the following book

Physics for Scientists and Engineers (with Physics NOW and InfoTrac),

Raymond A. Serway - Emeritus, James Madison University , Thomson Brooks/Cole © 2004, 6th Edition, 1296 pages

Problems

(1) An air-filled capacitor consists of two parallel plates, each with an area of 7.60 cm^2, separated by a distance of 1.80 mm. A 20.0-V potential difference is applied to these plates. Calculate (a) the electric field between the plates, (b) the surface charge density, (c) the capacitance, and (d) the charge on each plate. ***(problem 26.7)***

(a) $E = \dfrac{\Delta V}{d} = \dfrac{20}{1.8 \; x \; 10^{-3}} = 11.1 \; x \; 10^{3} \quad volt/m$

$$E = \frac{\sigma}{\varepsilon_o}$$

(b) $\sigma = E \;\; \varepsilon_o = 11.1 \; x \; 10^{3} \quad x \;\; 8.85 \; x \; 10^{-12}$

$$= 98.235 \;\; x \; 10^{-9} \quad C/m^2$$

(c) $C = \dfrac{\varepsilon_o \;\; A}{d} = \dfrac{8.85 x 10^{-12} \quad x \; 7.6 \; x 10^{-4}}{1.8 \;\; x \; 10^{-3}} = 37.367 \; x \; 10^{-13} \;\; C = 3.7367 \; pF$

(d) $Q = C \;\; \Delta V = 37.367 \; x 10^{-13} \quad x \;\; 20 = 74.73 \; x \; 10^{-12} = 74.73 \; pC$

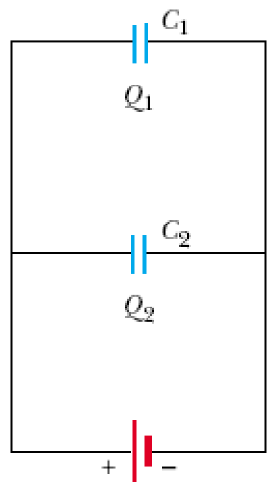
(2) When a potential difference of 150 V is applied to the plates of a parallel-plate capacitor, the plates carry a surface charge density of 30.0 nC/cm^2. What is the spacing between the plates? ***(problem26.9)***

$$\Delta V = E \; d = \frac{\sigma \;\; d}{\varepsilon_o}$$

$$d = \frac{\Delta V \quad \varepsilon_o}{\sigma} = \frac{150 \;\; x \;\; 8.85 \; x \; 10^{-12}}{30 \; x \; 10^{-9}}$$

$$= 44.25 \; x \; 10^{-3} \quad m$$

(3) Two capacitors, $C1 = 5.00\ \mu F$ and $C2 = 12.0\ \mu F$, are connected in parallel, and the resulting combination is connected to a 9.00-V battery. (a) What is the equivalent capacitance of the combination? What are (b) the potential difference across each capacitor and (c) the charge stored on each capacitor?



(a) $$\begin{aligned} C_{eq} &= C_1 + C_2 = 5\ x\ 10^{-6} + 12\ x\ 10^{-6} \\ &= 17\ x\ 10^{-6}\ F = 17\ \mu F \end{aligned}$$

(b) The individual potential differences across capacitors connected in parallel are the same and are equal to the potential difference applied across the combination, **So** $\Delta V_1 = \Delta V_2 = \Delta V = 9\ volt$

(c) $$\begin{aligned} Q_1 &= C_1\ x\ \Delta V = 5\ x\ 10^{-6}\ x\ 9 = 45\ x\ 10^{-6}\ C = 45\ \mu C \\ Q_2 &= C_2\ x\ \Delta V = 12\ x\ 10^{-6}\ x\ 9 = 108\ x\ 10^{-6}\ C = 108\ \mu C \end{aligned}$$

(4) Two capacitors when connected in parallel give an equivalent capacitance of 9.00 pF and give an equivalent capacitance of 2.00 pF when connected in series. What is the capacitance of each capacitor?

$$C_1 + C_2 = 9$$

$$\therefore\ C_1 = 9 - C_2 \qquad (1)$$

$$\frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{2}$$

$$\frac{1}{9 - C_2} + \frac{1}{C_2} = \frac{1}{2}$$

$$\frac{C_2 + 9 - C_2}{C_2\,(9 - C_2)} = \frac{1}{2}$$

$$9\ C_2 - C_2^2 = 18$$

$$C_2^2 - 9\ C + 18 = 0$$

$$(C_2 - 6)\,(C_2 - 3) = 0$$

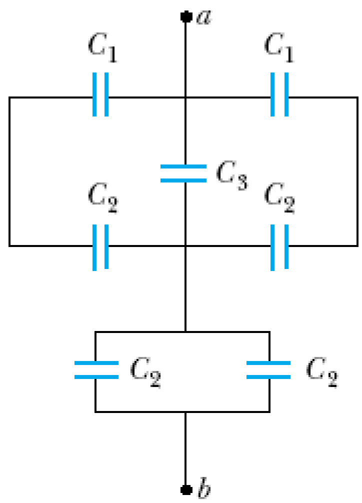
$$C_2 = 3\ pF \quad or\ C_2 = 6\ pF$$

$$and\ \ from\ Eq.\ (1)$$

$$C_1 = 6\ pF \quad or\ C_1 = 3\ pF$$

$$\therefore\ C_1 = 3\ pF \ \ and\ \ C_2 = 6\ pF$$

(5) Find the equivalent capacitance between points *a* and *b* for the group of capacitors connected as shown in the Figure . Take $C1 = 5.00\ \mu F$, $C2 = 10.0\ \mu F$, and $C3 = 2.00\ \mu F$.



$$C_s = \left(\frac{1}{5.00} + \frac{1}{10.0}\right)^{-1} = 3.33\ \mu F$$

$$C_{p1} = 2(3.33) + 2.00 = 8.66\ \mu F$$

$$C_{p2} = 2(10.0) = 20.0\ \mu F$$

$$C_{eq} = \left(\frac{1}{8.66} + \frac{1}{20.0}\right)^{-1} = \boxed{6.04\ \mu F}$$