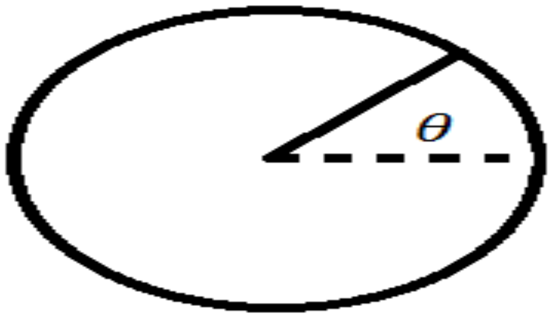
Rotational Motion Under Constant Angular Acceleration

For objects that rotate with a constant angular acceleration, there are a few equations, called the kinematic equations for angular motion, that you can use to determine the angular position or angular velocity of the object at any point in time.



θ = angular position
ω = angular velocity
α = angular acceleration

$$v_{av} \equiv \frac{x_f - x_i}{t_f - t_i} = \frac{\Delta x}{\Delta t}$$

$$\omega_{av} \equiv \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta \theta}{\Delta t}$$

Linear Motion with a Constant (Variables: x and v)	Rotational Motion About a Fixed Axis with α Constant (Variables: θ and ω)
$v = v_i + at$	$\omega = \omega_i + \alpha t$
$\Delta x = v_i t + \frac{1}{2}at^2$	$\Delta\theta = \omega_i t + \frac{1}{2}\alpha t^2$
$v^2 = v_i^2 + 2a\Delta x$	$\omega^2 = \omega_i^2 + 2\alpha\Delta\theta$

Example 10.1 Rotating wheel

A wheel rotates with a constant angular acceleration of $3.5\, rad/s^2$
If the angular speed of the wheel is 2.00 rad/s at t = 0,
(a) through what angle does the wheel rotate in 2.00 s?
(b) Through how many revolutions has the wheel turned during this time interval?
(c) What angular speed of the wheel at t=2.0 s ?

Example 2

A compact disc rotates from rest up to an angular speed of 31.4 rad/s in a time of 0.892 s.

(a) What is the angular acceleration of the disc, assuming the angular acceleration is uniform?

(b) Through what angle does the disc turn while coming up to speed?

(c) If the radius of the disc is 4.45 cm, find the tangential speed of a microbe riding on the rim of the disc when t = 0.892 s.

(d) What is the magnitude of the tangential acceleration of the microbe at the given time?

$$a_t = r\alpha$$

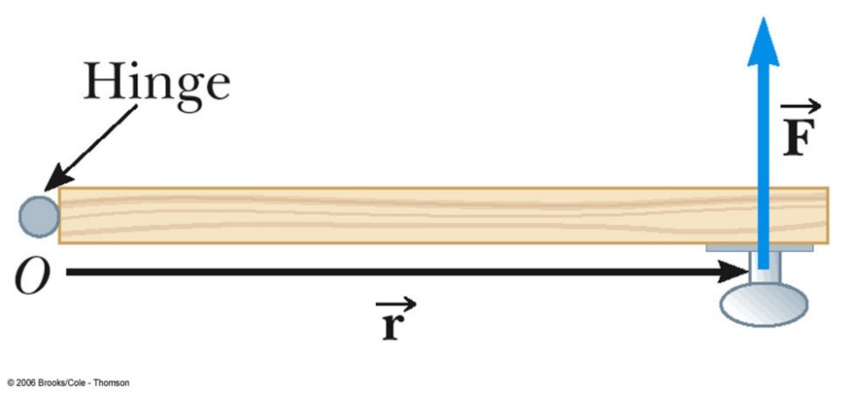
$$v_t = r\omega$$

Torque

- A torque is an action that causes objects to rotate.
- Torque is not the same thing as force.
- For rotational motion, the torque is what is most directly related to the motion, not the force.

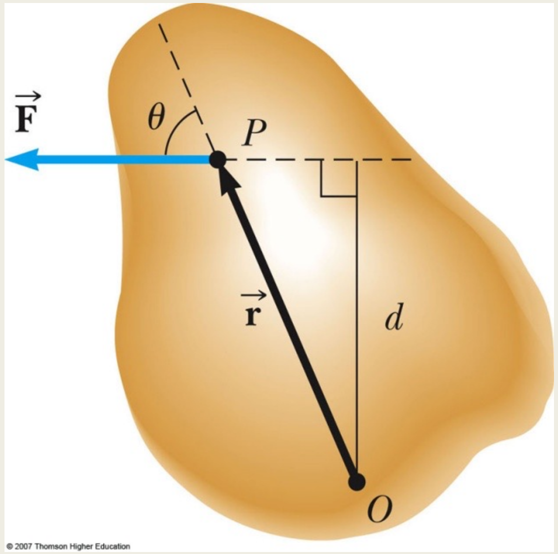
- Torque, τ, is the tendency of a force to rotate an object about some axis
- Let **F** be a force acting on an object, and let **r** be a position vector from a rotational center to the point of application of the force, with **F** perpendicular to **r**. The magnitude of the torque is given by

$$\tau = rF$$



- The SI units of torque are N·m
- Torque is a vector quantity
- Torque magnitude is given by

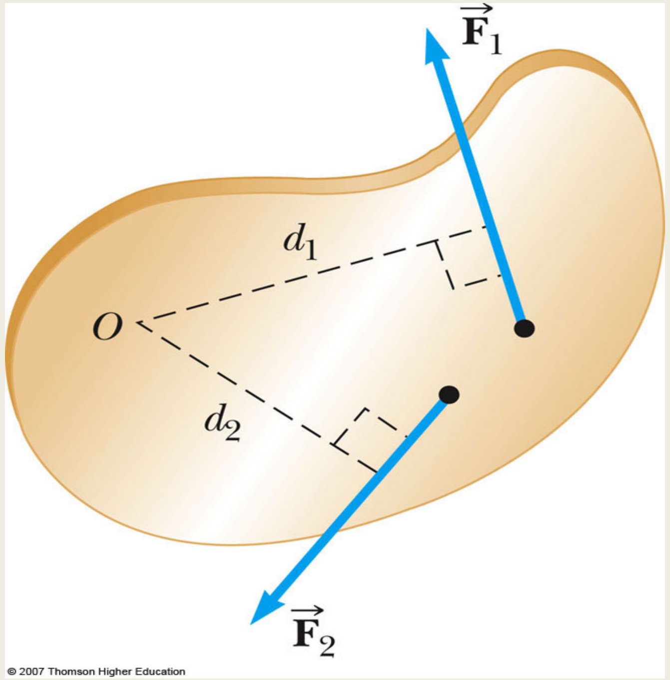
$$\tau = rF\sin\theta = Fd$$



- Torque will have direction
  - *If the turning tendency of the force is counterclockwise, the torque will be positive*
  - *If the turning tendency is clockwise, the torque will be negative*

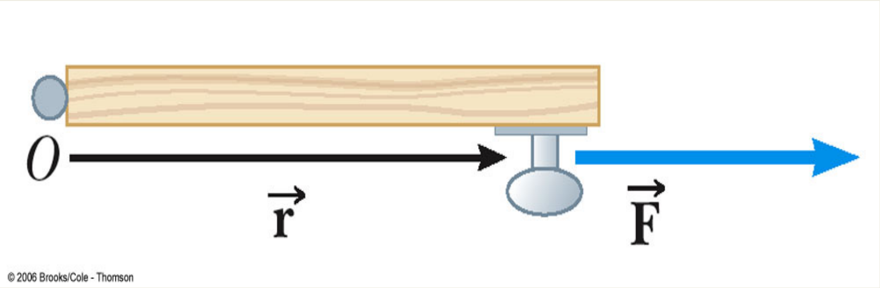
Net Torque

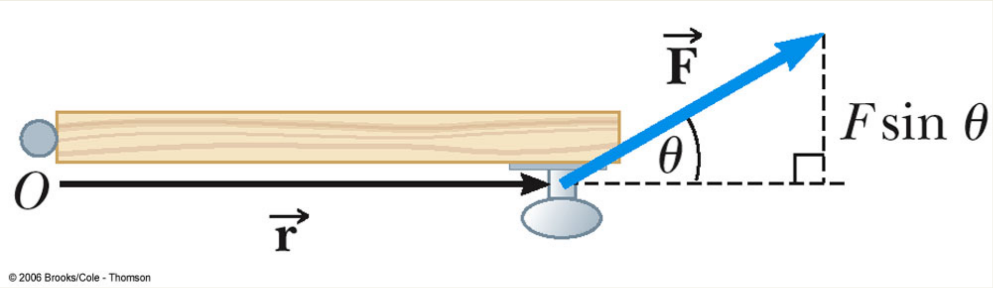
- The force will tend to cause a counterclockwise rotation about *O*
- The force will tend to cause a clockwise rotation about *O*
- $\Sigma\tau = \tau_1 + \tau_2 = F_1d_1 - F_2d_2$
- If $\Sigma\tau \neq 0$, starts rotating
- If $\Sigma\tau = 0$, rotation rate does not change



General Definition of Torque

- The applied force is not always perpendicular to the position vector
- The component of the force *perpendicular* to the object will cause it to rotate
- When the force is parallel to the position vector, no rotation occurs
- When the force is at some angle, the perpendicular component causes the rotation

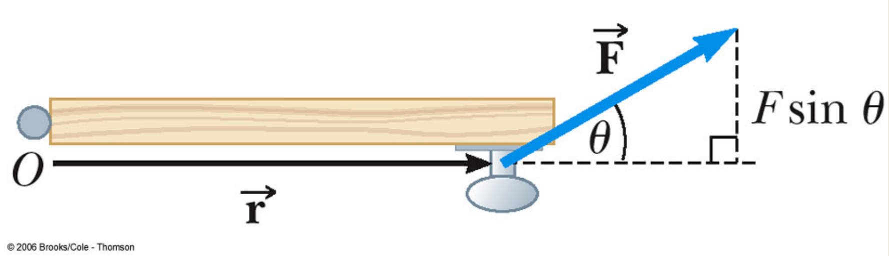




General Definition of Torque

- Let **F** be a force acting on an object, and let **r** be a position vector from a rotational center to the point of application of the force. The magnitude of the torque is given by

$$\tau = rF \sin\theta$$



- $\theta = 0^\circ$ or $\theta = 180^\circ$:

torque are equal to zero

- $\theta = 90^\circ$ or $\theta = 270^\circ$: magnitude of torque attain to the maximum

Periodic Motion

- Periodic motion is motion of an object that regularly returns to a given position after a fixed time interval

- A special kind of periodic motion occurs in mechanical systems when the force acting on the object is proportional to the position of the object relative to some equilibrium position

- If the force is always directed toward the equilibrium position, the motion is called simple harmonic motion

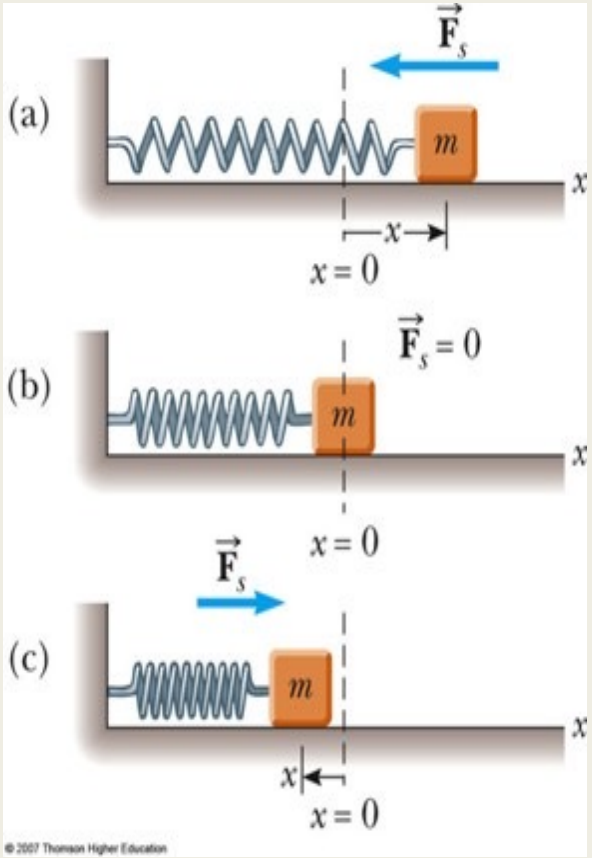
Motion of a Spring-Mass System

- A block of mass m is attached to a spring, the block is free to move on a frictionless horizontal surface
- Use the active figure to vary the initial conditions and observe the resultant motion
- When the spring is neither stretched nor compressed, the block is at the equilibrium position $x = 0$

Hooke's Law

Hooke's Law states $F_s = -kx$

- **F_s is the restoring force**
- It is always directed toward the equilibrium position
- Therefore, it is always opposite the displacement from equilibrium
- k is the force (spring) constant
- x is the displacement



Acceleration

The force described by Hooke's Law is the net force in Newton's Second Law

$$F_{\text{Hooke}} = F_{\text{Newton}}$$

$$-kx = ma_x$$

$$a_x = -\frac{k}{m}x$$

- The acceleration is proportional to the displacement of the block
- The direction of the acceleration is opposite the direction of the displacement from equilibrium
- An object moves with simple harmonic motion whenever its acceleration is proportional to its position and is oppositely directed to the displacement from equilibrium

Simple Harmonic Motion-Mathematical Representation

Model the block as a particle

- The representation will be particle in simple harmonic motion model
- Choose x as the axis along which the oscillation occurs

Acceleration

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

$$a = \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

We let

$$\omega^2 = \frac{k}{m}$$

Then $a = -\omega^2 x$

The particle motion is represented by the second order differential equation

$$\frac{d^2x}{dt^2} = -w^2 x$$

Simple Harmonic Motion – graphical Representation

A solution is $x(t) = A \cos (wt + \phi)$

A, w, ϕ are all constants

A cosine curve can be used to give physical significance to these constants

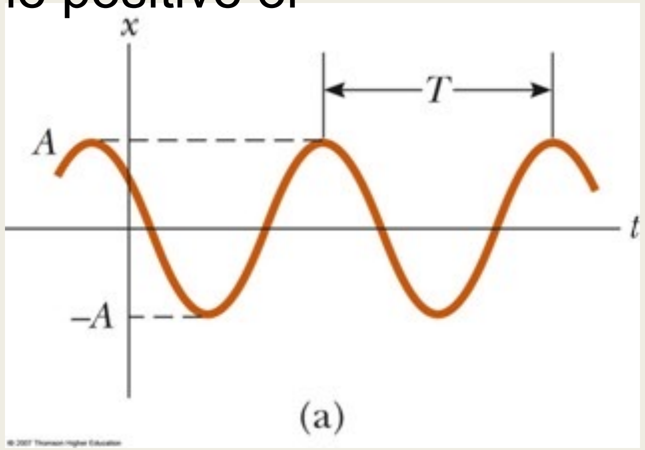
A is the amplitude of the motion

This is the maximum position of the particle in either the positive or negative direction

w is called the angular frequency

Units are rad/s

ϕ is the phase constant or the initial phase angle



(a)

Period

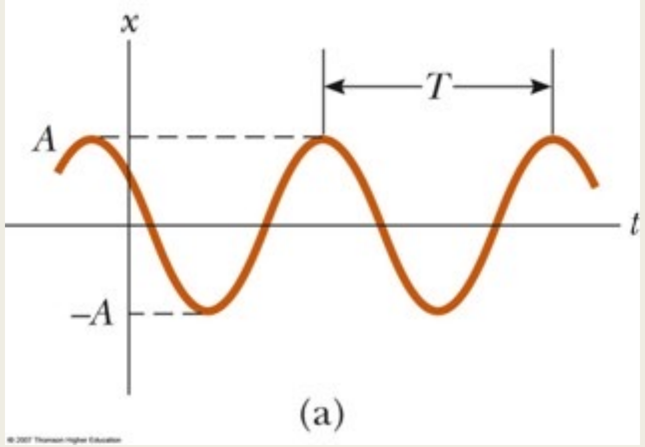
- ❖ The period, T, is the time interval required for the particle to go through one full cycle of its motion
- ❖ The values of x and v for the particle at time t equal the values of x and v at t + T because the phase increase by $2\pi\ rad$ in time interval of T
- ❖ $w(t+T) - wt = 2\pi$ then

$$T = \frac{2\pi}{\omega}$$

Frequency

- The inverse of the period is called the frequency
- The frequency represents the number of oscillations that the particle undergoes per unit time interval
- Units are cycles per second = hertz (Hz)

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$



Summary Equations – Period and Frequency

- The frequency and period equations can be rewritten to solve for w

$$\omega = 2\pi f = \frac{2\pi}{T}$$

- The period and frequency can also be expressed as:

$$T = 2\pi\sqrt{\frac{m}{k}} \qquad f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$\omega^2 = \frac{k}{m}$$

Motion Equations for Simple Harmonic Motion

Simple harmonic motion is one-dimensional and so directions can be denoted by + or - sign

$$x(t) = A\cos(\omega t + \phi)$$

$$v = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi)$$

$$a = \frac{d^2x}{dt^2} = -\omega^2 A \cos(\omega t + \phi)$$

Maximum Values of v and a

- Because the sine and cosine functions oscillate between ±1, we can easily find the maximum values of velocity and acceleration for an object in SHM

$$v_{max} = \omega A = \sqrt{\frac{k}{m}}\, A$$

$$a_{max} = \omega^2 A = \frac{k}{m}\, A$$

Energy of the SHM Oscillator

- Assume a spring-mass system is moving on a frictionless surface
- the total energy is constant
- The kinetic energy can be found by
- $K = \frac{1}{2} mv^2 = \frac{1}{2} mw^2A^2 \sin^2 (wt + \phi)$
- The elastic potential energy can be found by
- $U = \frac{1}{2} kx^{2} = \frac{1}{2} kA^2 \cos^2 (wt + \phi)$

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- $U = \frac{1}{2} kx^{2} = \frac{1}{2} kA^2 \cos^2 (wt + \phi)$

Energy of the SHM Oscillator

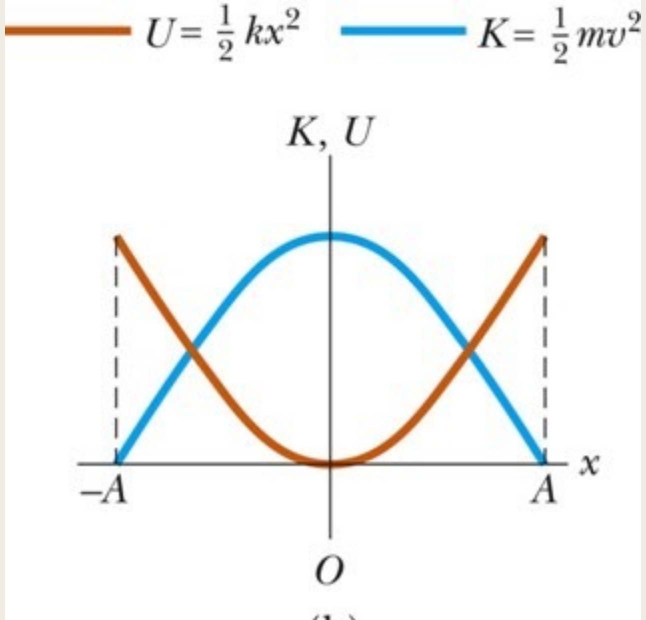
The total energy is $E = K + U = \frac{1}{2} kx^2 + \frac{1}{2} mv^2 = \frac{1}{2}kA^2$

- The total mechanical energy is constant
- The total mechanical energy is proportional to the square of the amplitude
- Energy is continuously being transferred between potential energy stored in the spring and the kinetic energy of the block
- Use the active figure to investigate the relationship between the motion and the energy

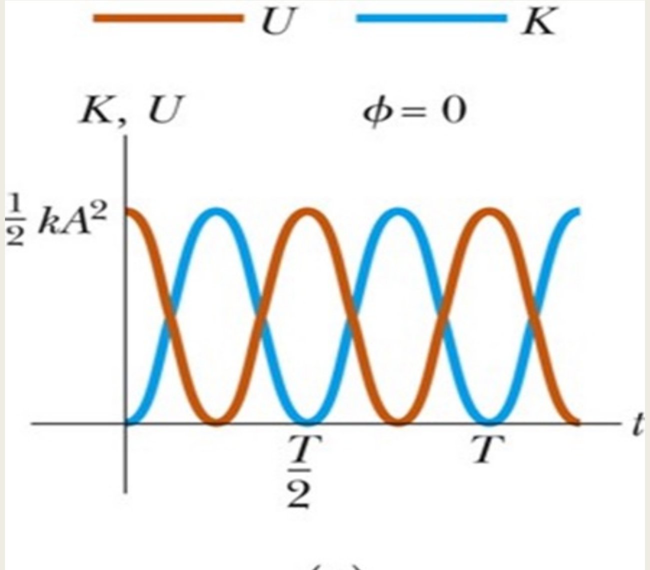
  - Eenergy of the SHM Oscillator,

As the motion continues, the exchange of energy also continues
Energy can be used to find the velocity

$$v = \pm\sqrt{\frac{k}{m}\left(A^2 - x^2\right)}$$
$$= \pm\omega^2\sqrt{A^2 - x^2}$$



(b)



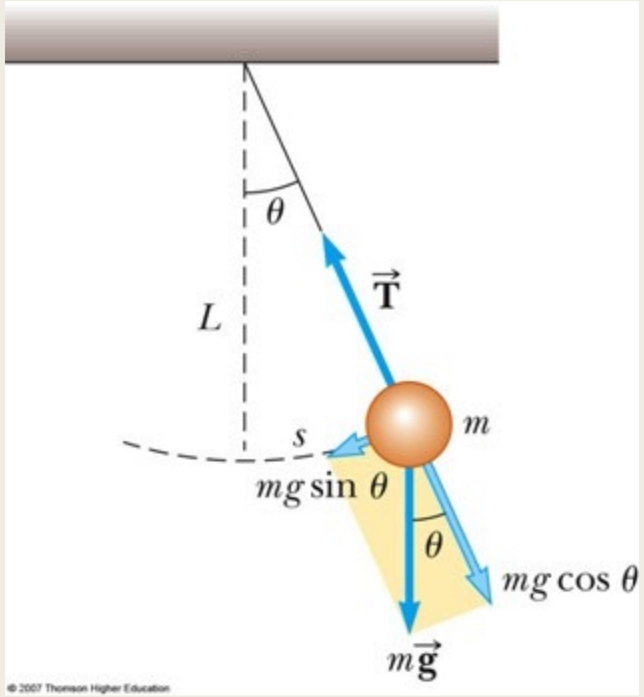
(a)

Simple Pendulum

- A simple pendulum also exhibits periodic motion
- The motion occurs in the vertical plane and is driven by gravitational force
- The motion is very close to that of the SHM oscillator
- If the angle is <10o

ϕ

- $\vec{T}$ The forces acting on the bob are the tension and the weight
- $m\vec{g}$ is the force exerted on the bob by the string
- is the gravitational force
- The tangential component of the gravitational force is a restoring force



- Simple Pendulum, In the tangential direction,

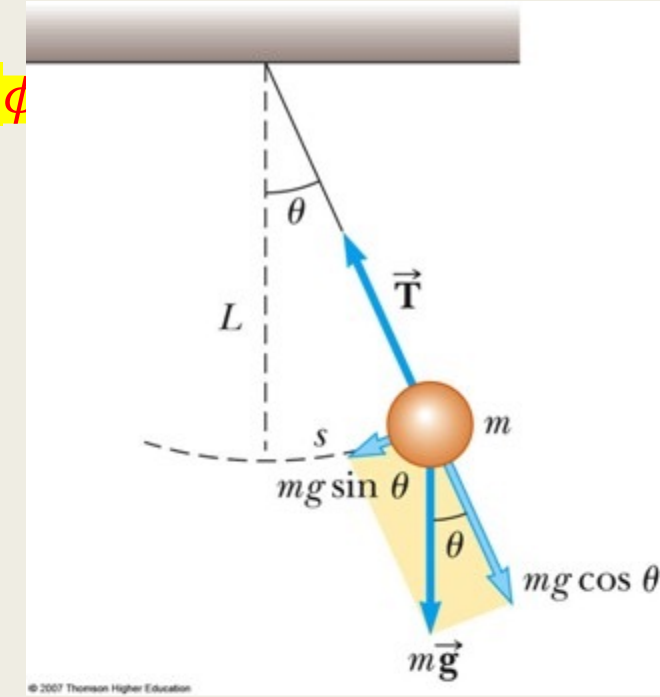
$$F_t = -mg\sin\theta = m\frac{d^2s}{dt^2}$$

- S= θ L
- The length, L, of the pendulum is constant, and for small values of θ , sin θ = θ
- $$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta = -\frac{g}{L}\theta$$
- Simple Pendulum, The function θ can be written as $x(t) = A\cos(wt + \phi$
- $\theta = \theta_{max}\cos(wt + \theta)$
- The angular frequency is

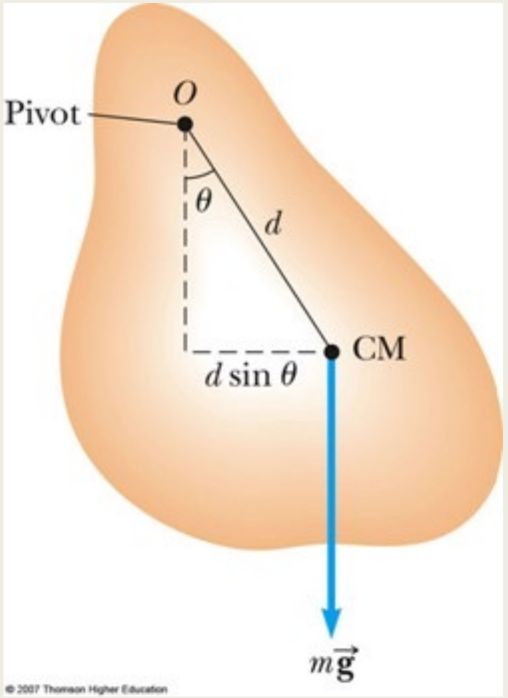
$$\omega = \sqrt{\frac{g}{L}}$$

- The period is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$



- Physical Pendulum
- If a hanging object oscillates about a fixed axis that does not pass through the center of mass and the object cannot be approximated as a particle, the system is called a physical pendulum
- It cannot be treated as a simple pendulum



Physical Pendulum,

The gravitational force provides a torque about an axis through O

The magnitude of the torque is *mgd sin θ*

I is the moment of inertia about the axis through O

From Newton second law,

$$-mgd\sin\theta = I\frac{d^2\theta}{dt^2}$$

Assuming *θ is small this become*

$$\frac{d^2\theta}{dt^2} = -\left(\frac{mgd}{I}\right)\theta = -\omega^2\theta$$

$$\omega = \sqrt{\frac{mgd}{I}}$$

EXAMPLE 15.1 A Block-Spring System

A 200-g block connected to a light spring for which the force constant is 5.00 N/m is free to oscillate on a horizontal, frictionless surface. The block is displaced 5.00 cm from equilibrium and released from rest as in Active Figure 15.6.

(A) Find the period of its motion.

SOLUTION

Conceptualize Study Figure 15.6 and imagine the block moving back and forth in simple harmonic motion once it is released. Set up an experimental model in the vertical direction by hanging a heavy object such as a stapler from a strong rubber band.

Categorize The block is modeled as a *particle in simple harmonic motion.*

Analyze

Use Equation 15.9 to find the angular frequency of the block–spring system:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{5.00\ \mathrm{N/m}}{200 \times 10^{-3}\ \mathrm{kg}}} = 5.00\ \mathrm{rad/s}$$

Use Equation 15.13 to find the period of the system:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{5.00\ \mathrm{rad/s}} = 1.26\ \mathrm{s}$$

EXAMPLE 15.3 **Oscillations on a Horizontal Surface**

A 0.500-kg cart connected to a light spring for which the force constant is 20.0 N/m oscillates on a horizontal, frictionless air track.

(A) Calculate the total energy of the system and the maximum speed of the cart if the amplitude of the motion is 3.00 cm.

SOLUTION

Conceptualize The system oscillates in exactly the same way as the block in Active Figure 15.10

Analyze Use Equation 15.21 to express the total energy of the oscillator system and equate it to the kinetic energy of the system when the cart is at $x = 0$:

$$E = \frac{1}{2}kA^2 = \frac{1}{2}mv_{max}^2$$

Solve for the maximum speed and substitute numerical values:

$$v_{max} = \sqrt{\frac{k}{m}}A = \sqrt{\frac{20.0\ \text{N/m}}{0.500\ \text{kg}}}(0.030\,0\ \text{m}) = 0.190\ \text{m/s}$$