Chapter 12

Equilibrium and Elasticity

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12-1 Equilibrium

Learning Objectives

12.01 Distinguish between equilibrium and static equilibrium.

12.02 Specify the four conditions for static equilibrium.

12.03 Explain center of gravity and how it relates to center of mass.

12.04 For a given distribution of particles, calculate the coordinates of the center of gravity and the center of mass.

12-1 Equilibrium

- We often want objects to be stable despite forces acting on them
- Consider a book resting on a table, a puck sliding with constant velocity, a rotating ceiling fan, a rolling bicycle wheel with constant velocity
- These objects have the characteristics that:

1. The linear momentum of the center of mass is constant
2. The angular momentum about the center of mass, or any other point, is constant

12-1 Equilibrium

- Such objects are in **equilibrium**

$$\vec{P} = \text{a constant} \quad \text{and} \quad \vec{L} = \text{a constant.}$$ **Eq. (12-1)**

- In this chapter we are largely concerned with objects that are not moving at all; $P = L = 0$
- These objects are in **static equilibrium**
- The only one of the examples from the previous page in static equilibrium is the book at rest on the table

12-1 Equilibrium

- As discussed in 8-3, if a body returns to static equilibrium after a slight displacement, it is in *stable* static equilibrium
- If a small displacement ends equilibrium, it is *unstable*
- Despite appearances, this rock is in stable static equilibrium, otherwise it would topple at the slightest gust of wind

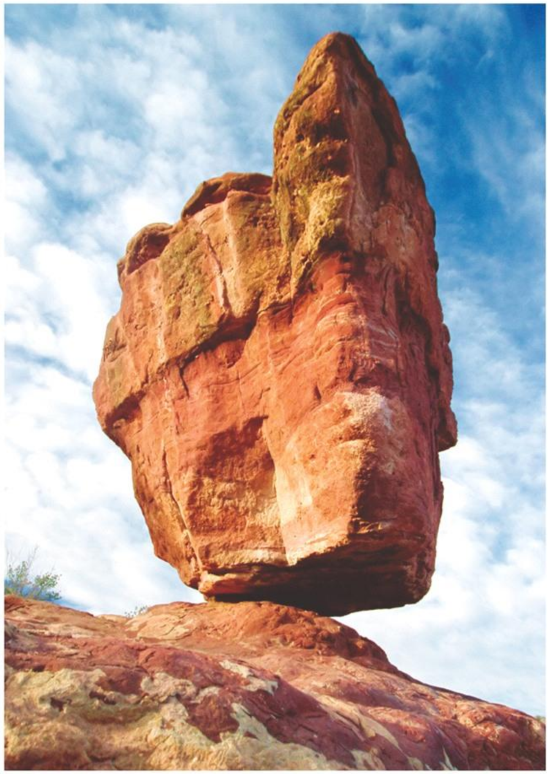

Figure 12-1

Kanwarjit Singh Boparai/Shutterstock

12-1 Equilibrium

- In part (a) of the figure, we have unstable equilibrium
- A small force to the right results in (b)
- In (c) equilibrium is stable, but push the domino so it passes the position shown in (a) and it falls
- The block in (d) is even more stable

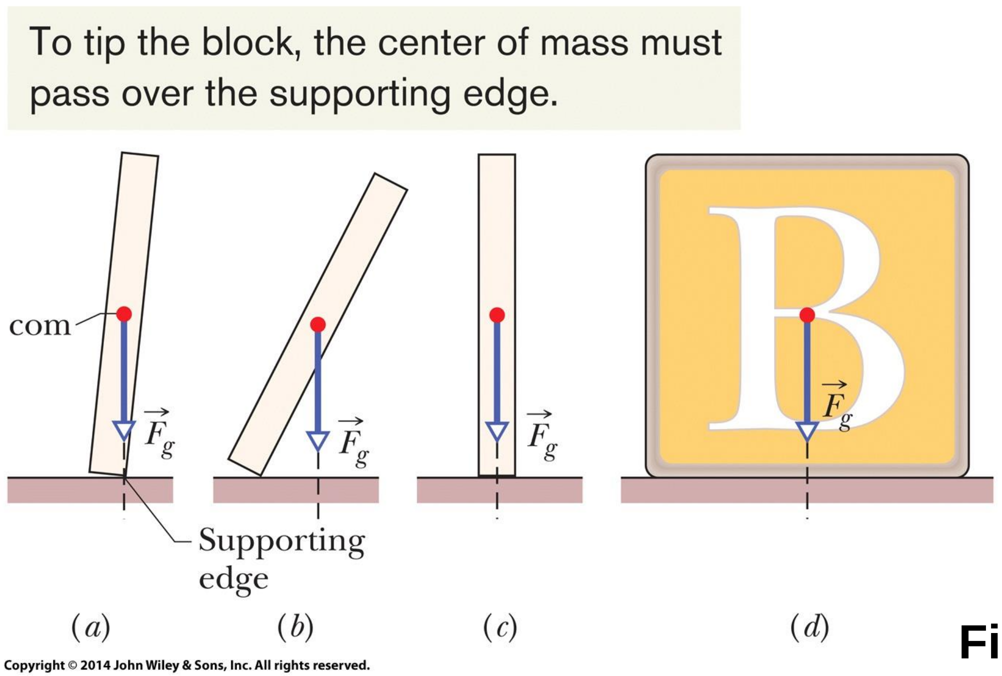

Figure 12-2

12-1 Equilibrium

- Requirements for equilibrium are given by Newton's second law, in linear and rotational form

$$\vec{F}_{\text{net}} = 0 \quad \text{(balance of forces).}$$ **Eq. (12-3)**

$$\vec{\tau}_{\text{net}} = 0 \quad \text{(balance of torques).}$$ **Eq. (12-5)**

- Therefore we have for equilibrium:

1. The vector sum of all the external forces that act on the body must be zero.

2. The vector sum of all external torques that act on the body, measured about *any* possible point, must also be zero.

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12-1 Equilibrium

- We often simplify matters by considering forces only in the *xy* plane, giving:

$$F_{\text{net},x} = 0 \quad \text{(balance of forces)},$$ **Eq. (12-7)**

$$F_{\text{net},y} = 0 \quad \text{(balance of forces)},$$ **Eq. (12-8)**

$$\tau_{\text{net},z} = 0 \quad \text{(balance of torques)}.$$ **Eq. (12-9)**

- Note that for static equilibrium we have the additional requirements that:

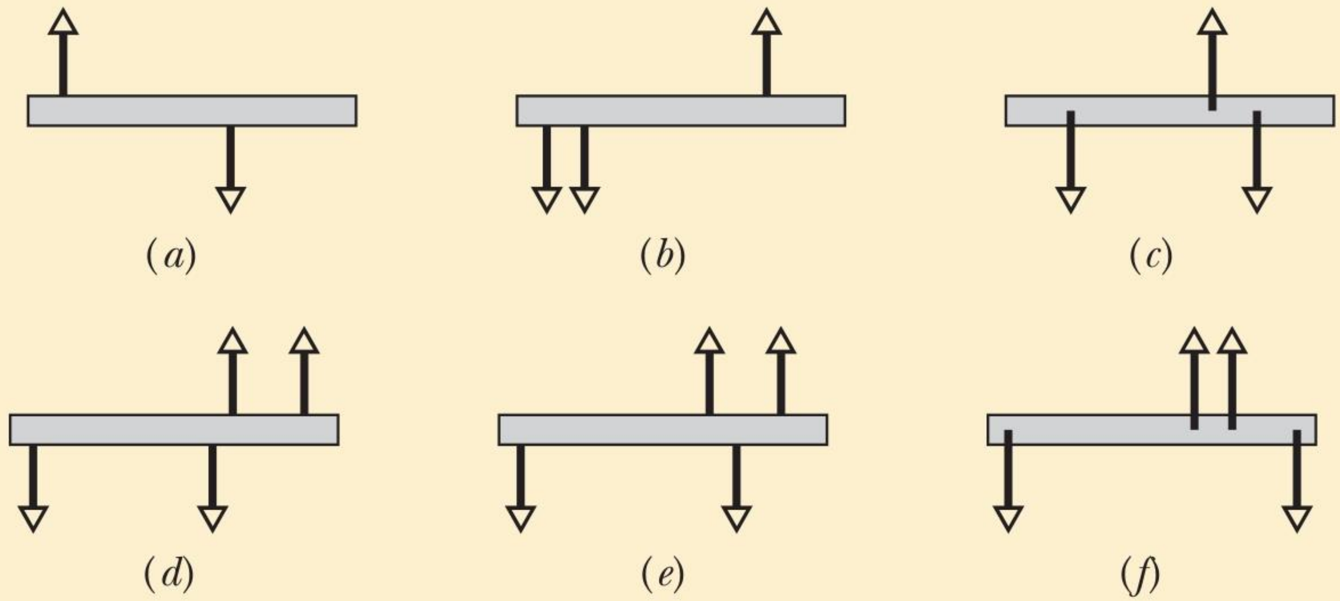
3. The linear momentum $\vec{P}$ of the body must be zero.

4. The angular momentum of the body **L** must be zero.

12-1 Equilibrium

Checkpoint 1

The figure gives six overhead views of a uniform rod on which two or more forces act perpendicularly to the rod. If the magnitudes of the forces are adjusted properly (but kept nonzero), in which situations can the rod be in static equilibrium?



Answer: (c), (e), (f)

12-1 Equilibrium

- The gravitational force on a body is the sum of gravitational forces acting on individual elements (atoms) of the body
- We can simplify this by saying:

The gravitational force $\vec{F}_g$ on a body effectively acts at a single point, called the **center of gravity** (cog) of the body.

- Until now we have assumed that the gravitational force acts at the center of mass
- This is approximately true for the everyday case:

If $\vec{g}$ is the same for all elements of a body, then the body's center of gravity (cog) is coincident with the body's center of mass (com).

12-1 Equilibrium

- We can show this by considering a sum of torques on each element vs. the torque caused by the gravitational force at the cog

$$x_{\text{cog}} \sum F_{gi} = \sum x_i F_{gi}.$$

- Substitute $m_i g_i$ for F_{gi}
- Cancel g (= g_i for all i)and divide by the total mass, leaving:

$$x_{\text{cog}} = \frac{1}{M} \sum x_i m_i. \quad \textbf{Eq. (12-16)}$$

- The term on the right is the com

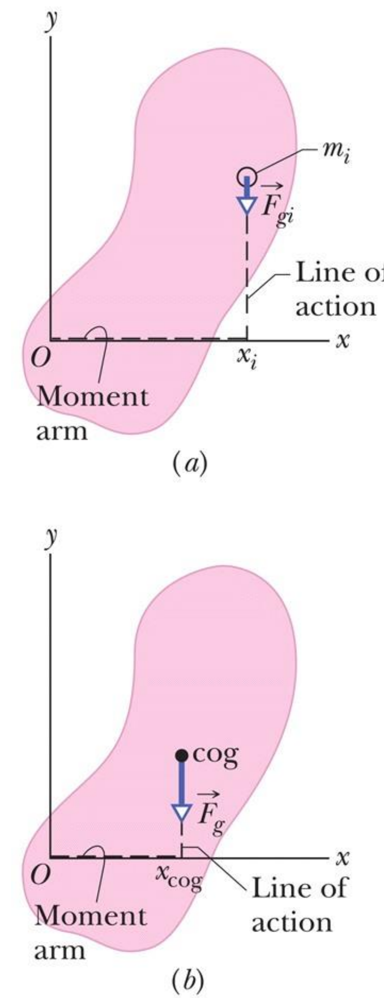


Figure 12-4

12-2 Some Examples of Static Equilibrium

Learning Objectives

12.05 Apply the force and torque conditions for static equilibrium.

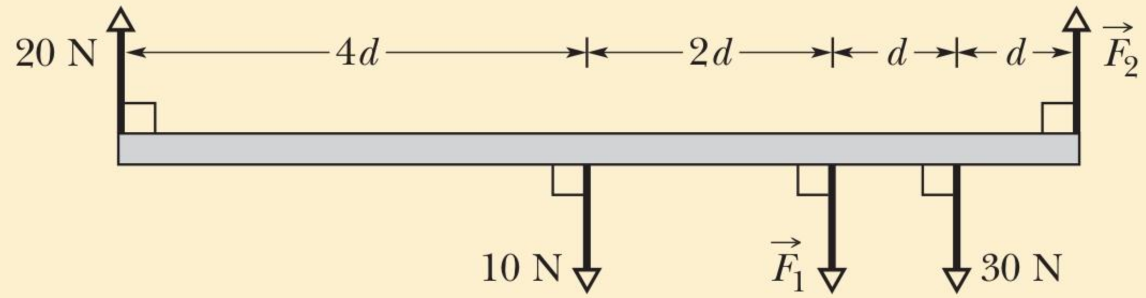
12.06 Identify that a wise choice about the placement of the origin (about which to calculate torques) can simplify the calculations by eliminating one or more unknown forces from the torque equation.

12-2 Some Examples of Static Equilibrium

- This section consists of example problems, for forces in the *xy* plane

Checkpoint 2

The figure gives an overhead view of a uniform rod in static equilibrium. (a) Can you find the magnitudes of unknown forces $\vec{F}_1$ and $\vec{F}_2$ by balancing the forces? (b) If you wish to find the magnitude of force $\vec{F}_2$ by using a balance of torques equation, where should you place a rotation axis to eliminate $\vec{F}_1$ from the equation? (c) The magnitude of $\vec{F}_2$ turns out to be 65 N. What then is the magnitude of $\vec{F}_1$?



Answer: (a) No (b) place the rotation axis at the location where F1 is applied to the beam (c) 45 N

12-2 Some Examples of Static Equilibrium

Example Balancing a horizontal beam

- M = 2.7 kg, m = 1.8 kg
- Set rotation axis at x = 0
- Sum torques
- $\frac{1}{4} Mg\, L + \frac{1}{2} mgL = F_r L$

  so F_r = 15 N

- Balance vertical forces

$$F_l = (M + m)g - F_r$$
$$= 29\ \text{N}$$

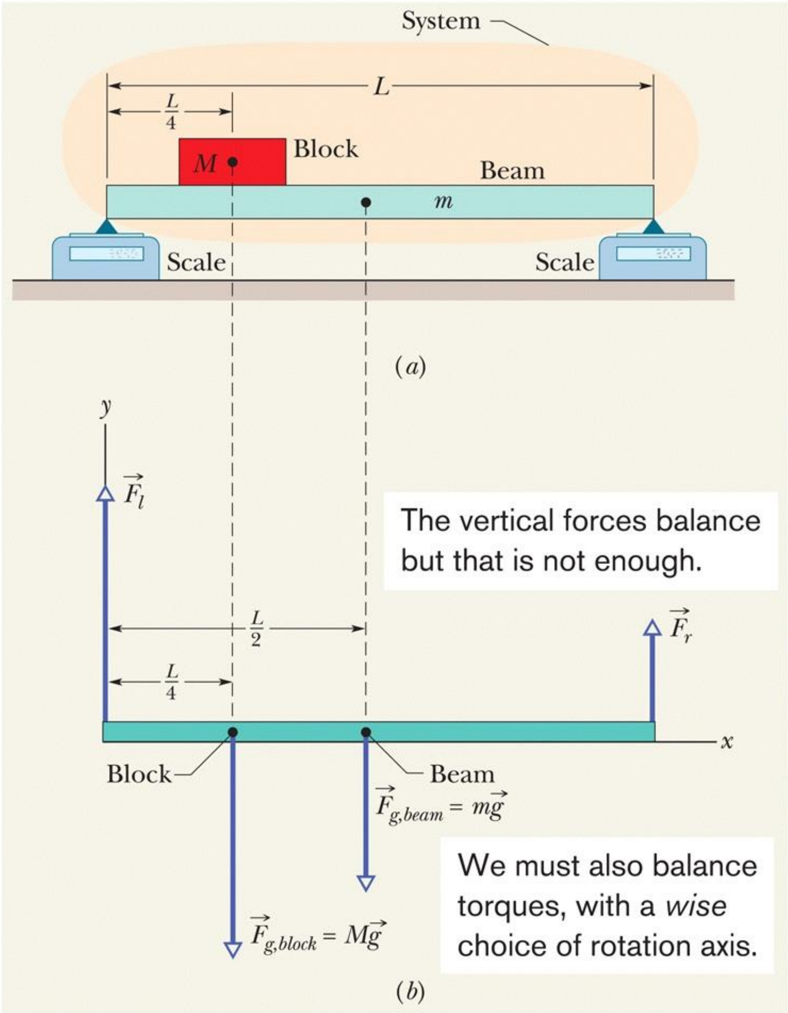


Figure 12-5

12-2 Some Examples of Static Equilibrium

Example Balancing a leaning beam:

Find the tension in the cable, in the rope and the size of force *F* by the hinge.

M = 430 kg, m = 85 kg, a = 1.9 m, b = 2.5 m

- Set rotation axis at $x = 0$, $y = 0$
- Sum torques (using $T_r = Mg$)
  - $a\,T_c - b\,T_r - \frac{1}{2}\,b\,mg = 0$
    $T_c = 6100$ N
- Balance forces
  - $F_h = T_c = 6100$ N
    $F_v = (m + M)\,g = 5050$ N
    $F = 7900$ N

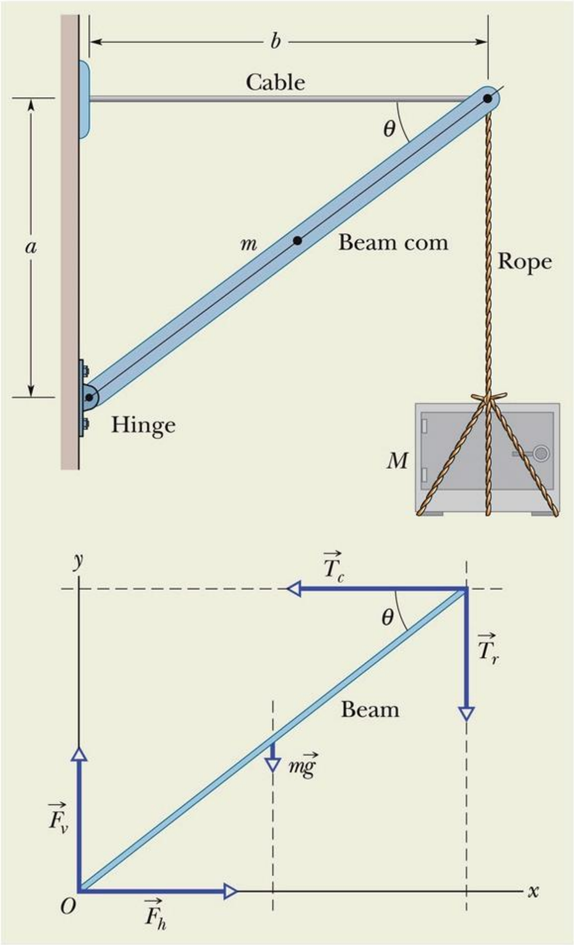


Figure 12-6

12-3 Elasticity

Learning Objectives

12.07 Explain what an indeterminate situation is.

12.08 For tension and compression, apply the equation that relates stress to strain and Young's modulus.

12.09 Distinguish between yield strength and ultimate strength.

12.10 For shearing, apply the equation that relates stress to strain and the shear modulus.

12.11 For hydraulic stress, apply the equation that relates fluid pressure to strain and the bulk modulus.

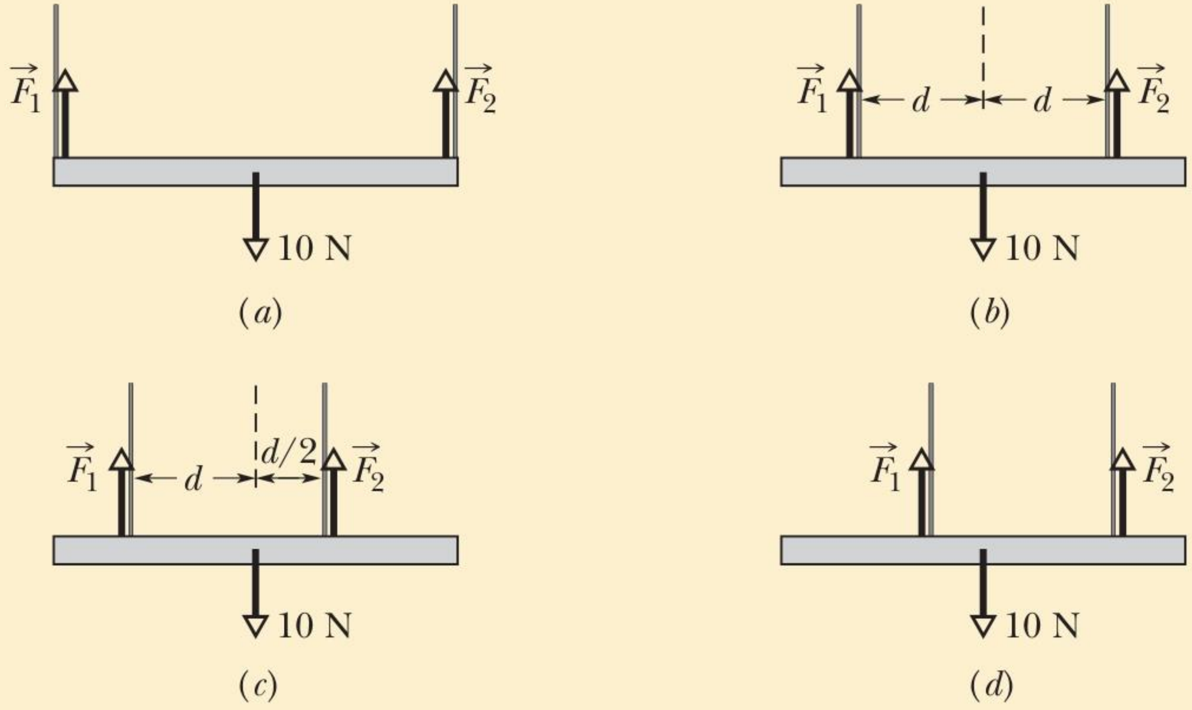
12-3 Elasticity

- For problems in the *xy* plane we have 3 independent equations
- Therefore we can solve for 3 unknowns
- If we have more unknown forces, we cannot solve for them and the situation is **indeterminate**
- This assumes that bodies are rigid and do not deform (there are no such bodies)
- With some knowledge of elasticity, we can solve more problems

12-3 Elasticity

Checkpoint 3

A horizontal uniform bar of weight 10 N is to hang from a ceiling by two wires that exert upward forces $\vec{F}_1$ and $\vec{F}_2$ on the bar. The figure shows four arrangements for the wires. Which arrangements, if any, are indeterminate (so that we cannot solve for numerical values of $\vec{F}_1$ and $\vec{F}_2$)?



Answer: (d)

12-3 Elasticity

- All rigid bodies are partially **elastic**, meaning we can change their dimensions by applying forces
- A **stress**, deforming force per unit area, produces a **strain**, or unit deformation
- There are 3 main types of stress:
  - Tensile (a), Shearing (b), Hydraulic ©

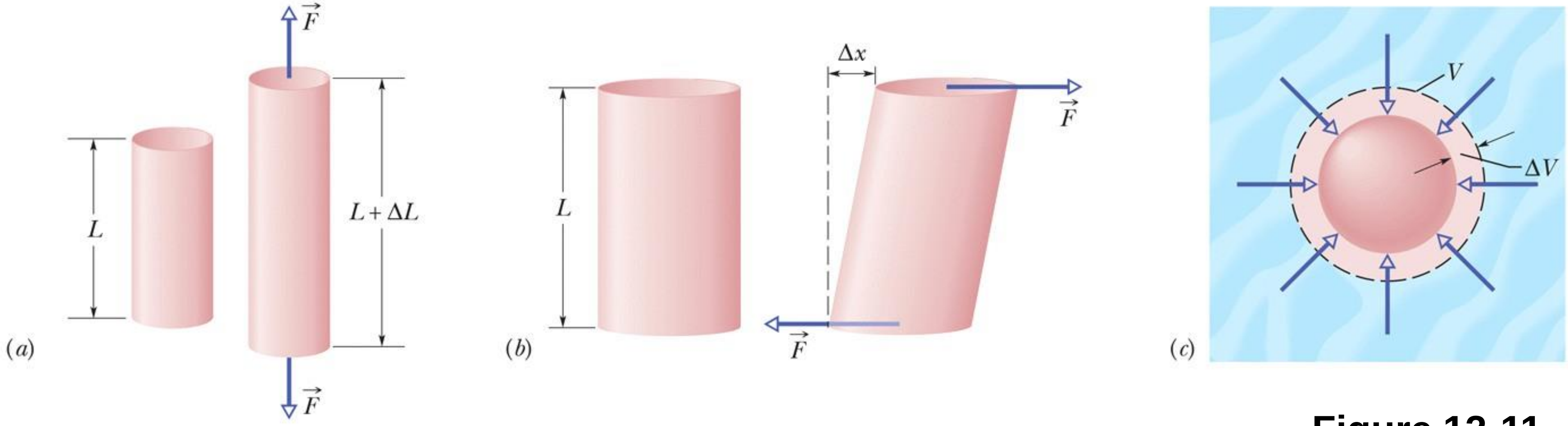


Figure 12-11

12-3 Elasticity

- Stress and strain are proportional in the elastic range
- Related by the **modulus of elasticity**:

$$\text{stress} = \text{modulus} \times \text{strain}. \qquad \textbf{Eq. (12-22)}$$

- As stress increases, eventually a **yield strength** is reached and the material deforms permanently
- At the **ultimate strength**, the material breaks

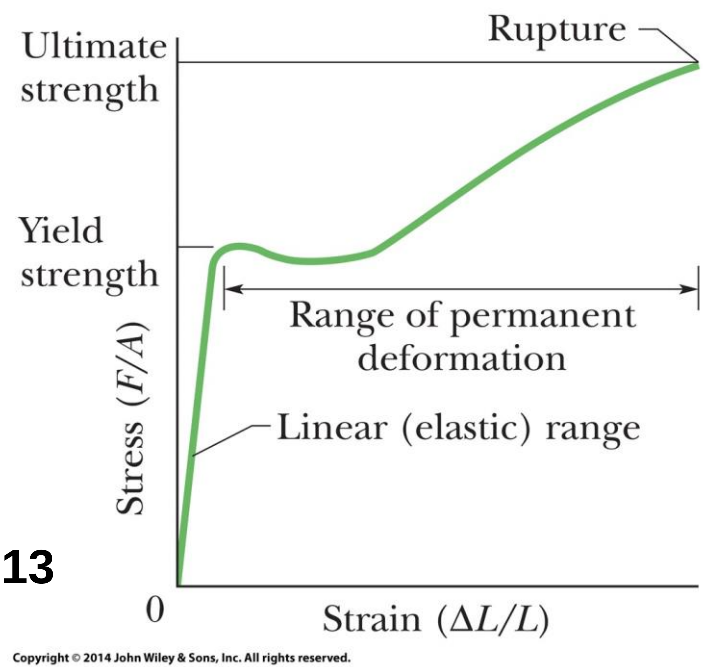


Figure 12-13

12-3 Elasticity

- In simple tension/compression, stress is *F*/*A*
- The strain is the dimensionless quantity *ΔL*/*L*
- **Young's modulus**, *E*, used for tension/compression

$$\frac{F}{A} = E\,\frac{\Delta L}{L}. \qquad \textbf{Eq. (12-23)}$$

- Note that many materials have very different tensile and compressive strengths, despite the same modulus being used for both
- E.g., concrete: high compressive strength, very low tensile strength

12-3 Elasticity

- **Shear modulus**, *G*, used for shearing

$$\frac{F}{A} = G\frac{\Delta x}{L}.$$ **Eq. (12-24)**

- Δx is along a different axis than L
- **Bulk modulus**, *B*, used for hydraulic compression

$$p = B\frac{\Delta V}{V}.$$ **Eq. (12-25)**

- Relates pressure to volume change

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12-3 Elasticity

- The table shows some elastic properties for common materials, for comparison purposes

Table 12-1 **Some Elastic Properties of Selected Materials of Engineering Interest**

Material	Density ρ (kg/m^3)	Young's Modulus E ($10^9\ N/m^2$)	Ultimate Strength S_u ($10^6\ N/m^2$)	Yield Strength S_y ($10^6\ N/m^2$)
Steel[a]	7860	200	400	250
Aluminum	2710	70	110	95
Glass	2190	65	50[b]	—
Concrete[c]	2320	30	40[b]	—
Wood[d]	525	13	50[b]	—
Bone	1900	9[b]	170[b]	—
Polystyrene	1050	3	48	—

[a]Structural steel (ASTM-A36). [b]In compression.
[c]High strength [d]Douglas fir.

Table 12-1

Sample Problem 12.05 Stress and strain of elongated rod

One end of a steel rod of radius $R = 9.5$ mm and length $L = 81$ cm is held in a vise. A force of magnitude $F = 62$ kN is then applied perpendicularly to the end face (uniformly across the area) at the other end, pulling directly away from the vise. What are the stress on the rod and the elongation ΔL and strain of the rod?

KEY IDEAS

(1) Because the force is perpendicular to the end face and uniform, the stress is the ratio of the magnitude F of the force to the area A. The ratio is the left side of Eq. 12-23. (2) The elongation ΔL is related to the stress and Young's modulus E by Eq. 12-23 ($F/A = E\,\Delta L/L$). (3) Strain is the ratio of the elongation to the initial length L.

Calculations: To find the stress, we write

$$\text{stress} = \frac{F}{A} = \frac{F}{\pi R^2} = \frac{6.2 \times 10^4\ \text{N}}{(\pi)(9.5 \times 10^{-3}\ \text{m})^2}$$

$$= 2.2 \times 10^8\ \text{N/m}^2. \qquad \text{(Answer)}$$

The yield strength for structural steel is 2.5×10^8 N/m², so this rod is dangerously close to its yield strength.

We find the value of Young's modulus for steel in Table 12-1. Then from Eq. 12-23 we find the elongation:

$$\Delta L = \frac{(F/A)L}{E} = \frac{(2.2 \times 10^8\ \text{N/m}^2)(0.81\ \text{m})}{2.0 \times 10^{11}\ \text{N/m}^2}$$

$$= 8.9 \times 10^{-4}\ \text{m} = 0.89\ \text{mm}. \qquad \text{(Answer)}$$

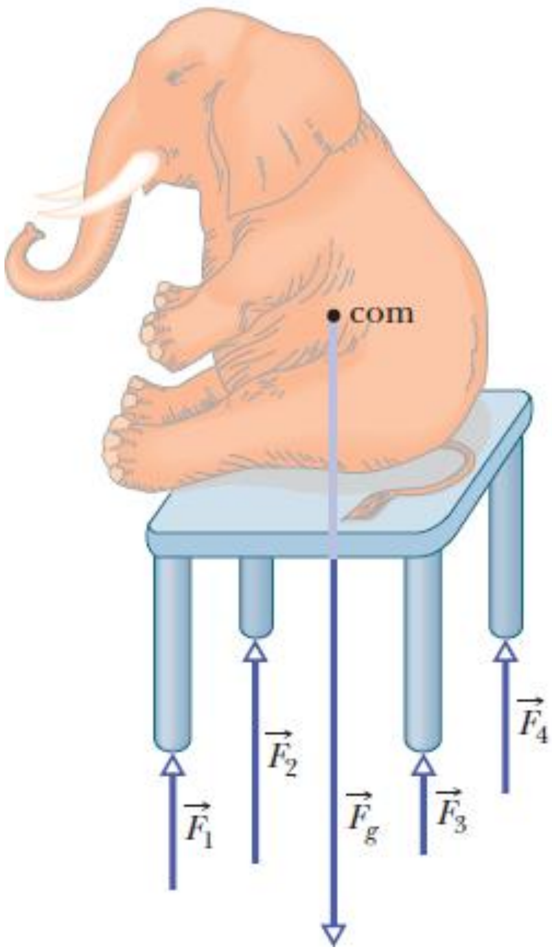
For the strain, we have

$$\frac{\Delta L}{L} = \frac{8.9 \times 10^{-4}\ \text{m}}{0.81\ \text{m}}$$

$$= 1.1 \times 10^{-3} = 0.11\%. \qquad \text{(Answer)}$$

Sample Problem 12.06 Balancing a wobbly table

A table has three legs that are 1.00 m in length and a fourth leg that is longer by $d = 0.50$ mm, so that the table wobbles slightly. A steel cylinder with mass $M = 290$ kg is placed on the table (which has a mass much less than M) so that all four legs are compressed but unbuckled and the table is level but no longer wobbles. The legs are wooden cylinders with cross-sectional area $A = 1.0\ \text{cm}^2$; Young's modulus is $E = 1.3 \times 10^{10}\ \text{N/m}^2$. What are the magnitudes of the forces on the legs from the floor?



Each of the short legs must be compressed by the same amount (call it ΔL_3) and thus by the same force of magnitude F_3. The single long leg must be compressed by a larger amount ΔL_4 and thus by a force with a larger magnitude F_4. In other words, for a level tabletop, we must have

$$\Delta L_4 = \Delta L_3 + d. \qquad (12\text{-}26)$$

mation gives us

$$\frac{F_4 L}{AE} = \frac{F_3 L}{AE} + d. \tag{12-27}$$

We cannot solve this equation because it has two unknowns, F_4 and F_3.

To get a second equation containing F_4 and F_3, we can use a vertical y axis and then write the balance of vertical forces ($F_{\text{net},y} = 0$) as

$$3F_3 + F_4 - Mg = 0, \tag{12-28}$$

$$\begin{aligned} F_3 &= \frac{Mg}{4} - \frac{dAE}{4L} \\ &= \frac{(290\ \text{kg})(9.8\ \text{m/s}^2)}{4} \\ &\quad - \frac{(5.0 \times 10^{-4}\ \text{m})(10^{-4}\ \text{m}^2)(1.3 \times 10^{10}\ \text{N/m}^2)}{(4)(1.00\ \text{m})} \\ &= 548\ \text{N} \approx 5.5 \times 10^2\ \text{N}. \end{aligned} \qquad \text{(Answer)}$$

From Eq. 12-28 we then find

$$\begin{aligned} F_4 &= Mg - 3F_3 = (290\ \text{kg})(9.8\ \text{m/s}^2) - 3(548\ \text{N}) \\ &\approx 1.2\ \text{kN}. \end{aligned} \qquad \text{(Answer)}$$

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12 Summary

Static Equilibrium

$$\vec{F}_{net} = 0 \quad \text{(balance of forces).}$$

Eq. (12-3)

$$\vec{\tau}_{net} = 0 \quad \text{(balance of torques).}$$

Eq. (12-5)

Center of Gravity

- If the gravitational acceleration is the same for all elements of the body, the cog is at the com.

Elastic Moduli

- Three elastic moduli
- Strain: fractional length change
- Stress: force per unit area

$$\text{stress} = \text{modulus} \times \text{strain}.$$

Eq. (12-22)

Tension and Compression

- *E* is Young's modulus

$$\frac{F}{A} = E\,\frac{\Delta L}{L}.$$

Eq. (12-23)

12 Summary

Shearing

- *G* is the shear modulus

$$\frac{F}{A} = G\frac{\Delta x}{L}.$$ **Eq. (12-24)**

Hydraulic Stress

- *B* is the bulk modulus

$$p = B\frac{\Delta V}{V}.$$ **Eq. (12-25)**