Chapter 10

Rotation

10-1 Rotational Varibles

Learning Objectives

10.01 Identify that if all parts of a body rotate around a fixed axis locked together, the body is a rigid body.

10.02 Identify that the angular position of a rotating rigid body is the angle that an internal reference line makes with a fixed, external reference line.

10.03 Apply the relationship between angular displacement and the initial and final angular positions.

10.04 Apply the relationship between average angular velocity, angular displacement, and the time interval for that displacement.

10.05 Apply the relationship between average angular acceleration, change in angular velocity, and the time interval for that change.

10.06 Identify that counterclockwise motion is in the positive direction and clockwise motion is in the negative direction.

10.07 Given angular position as a *function of time*, calculate the instantaneous angular velocity at any particular time and the average angular velocity between any two particular times.

10.08 Given a *graph* of angular position versus time, determine the instantaneous angular velocity at a particular time and the average angular velocity between any two particular times.

10.09 Identify instantaneous angular speed as the magnitude of the instantaneous angular velocity.

10.10 Given angular velocity as a *function of time*, calculate the instantaneous angular acceleration at any particular time and the average angular acceleration between any two particular times.

10.11 Given a *graph* of angular velocity versus time, determine the instantaneous angular acceleration at any particular time and the average angular acceleration between any two particular times.

10.12 Calculate a body's change in angular velocity by integrating its angular acceleration function with respect to time.

10.13 Calculate a body's change in angular position by integrating its angular velocity function with respect to time.

10-1 Rotational Variables

- We now look at motion of **rotation**
- We will find the same laws apply
- But we will need new quantities to express them
  - Torque
  - Rotational inertia
- A **rigid body** rotates as a unit, locked together
- We look at rotation about a **fixed axis**
- These requirements exclude from consideration:
  - The Sun, where layers of gas rotate separately
  - A rolling bowling ball, where rotation and translation occur

10-1 Rotational Variables

- The fixed axis is called the **axis of rotation**
- Figs 10-2, 10-3 show a *reference line*
- The **angular position** of this line (and of the object) is taken relative to a fixed direction, the **zero angular position**

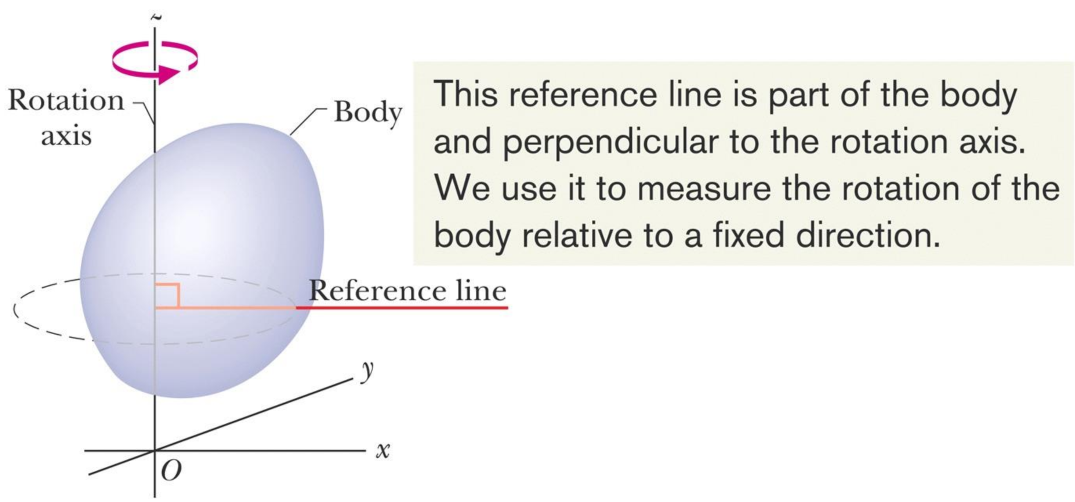


Figure 10-2

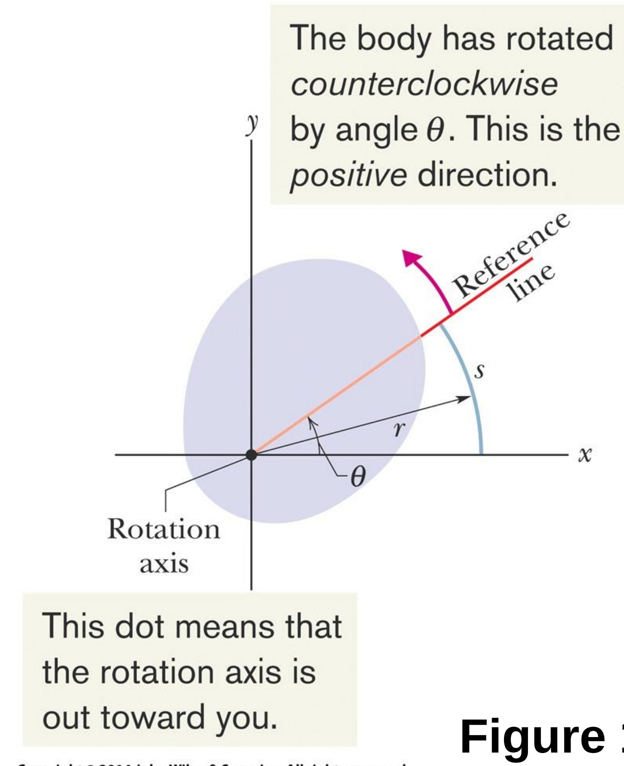


Figure 10-3

10-1 Rotational Variables

- Measure using **radians** (rad): dimensionless

$$\theta = \frac{s}{r} \quad \text{(radian measure).}$$ **Eq. (10-1)**

$$1 \text{ rev} = 360° = \frac{2\pi r}{r} = 2\pi \text{ rad},$$ **Eq. (10-2)**

- Do not reset θ to zero after a full rotation
- We know all there is to know about the kinematics of rotation if we have $\theta(t)$ for an object
- Define angular displacement as:

$$\Delta\theta = \theta_2 - \theta_1.$$ **Eq. (10-4)**

10-1 Rotational Variables

- "*Clocks are negative*":

An angular displacement in the counterclockwise direction is positive, and one in the clockwise direction is negative.

Checkpoint 1

A disk can rotate about its central axis like a merry-go-round. Which of the following pairs of values for its initial and final angular positions, respectively, give a negative angular displacement: (a) -3 rad, $+5$ rad, (b) -3 rad, -7 rad, (c) 7 rad, -3 rad?

Answer: Choices (b) and (c)

10-1 Rotational Variables

- **Average angular velocity**: angular displacement during a time interval

$$\omega_{\text{avg}} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}, \qquad \textbf{Eq. (10-5)}$$

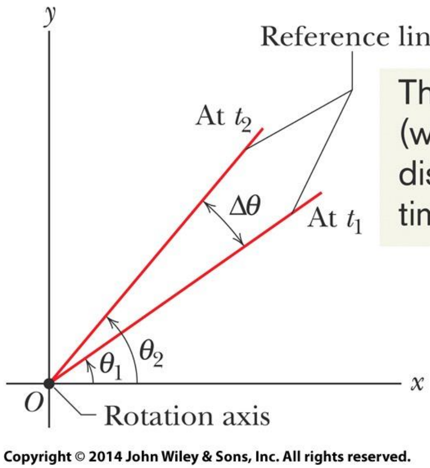
- **Instantaneous angular velocity**: limit as $\Delta t \rightarrow 0$

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}. \qquad \textbf{Eq. (10-6)}$$

- If the body is rigid, these equations hold for all points on the body
- Magnitude of angular velocity = **angular speed**

10-1 Rotational Variables

- Figure 10-4 shows the values for a calculation of average angular velocity



This change in the angle of the reference line (which is part of the body) is equal to the angular displacement of the body itself during this time interval.

Figure 10-4

- **Average angular acceleration**: angular velocity change during a time interval

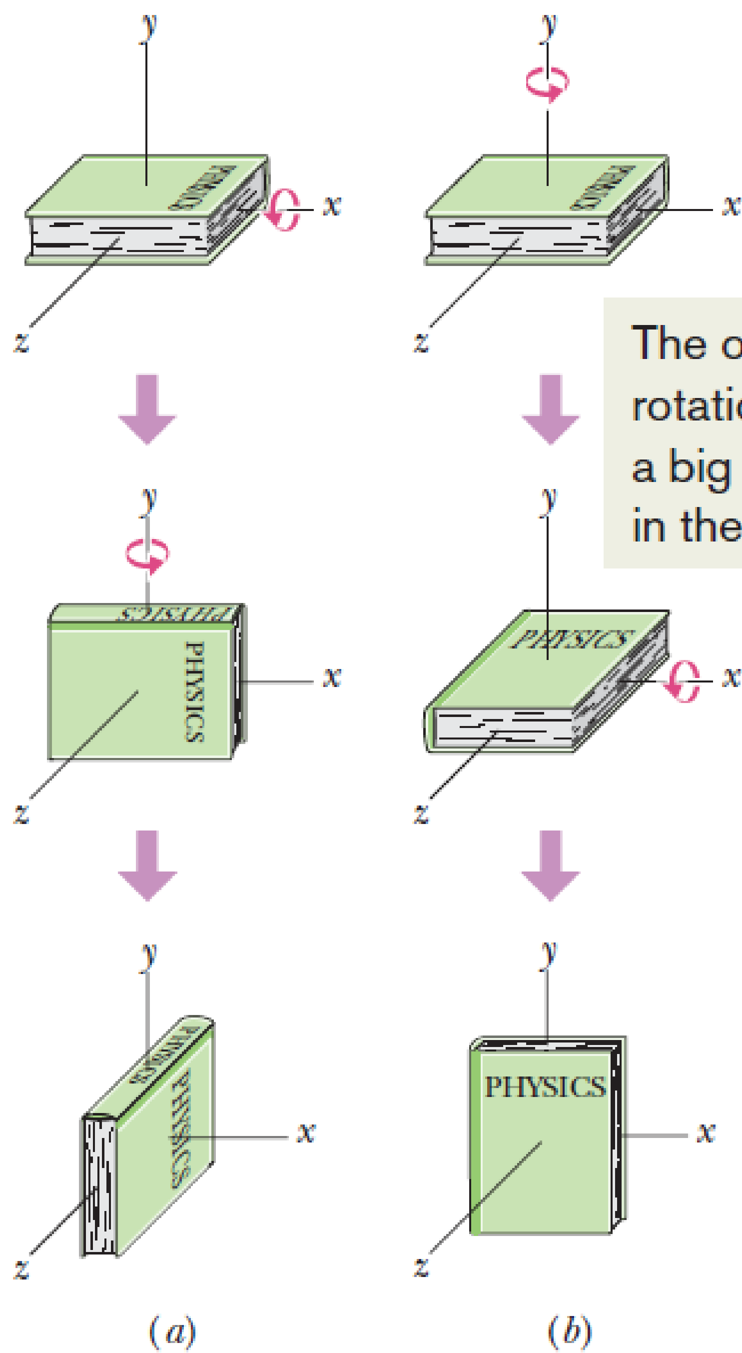
$$\alpha_{avg} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t},$$ **Eq. (10-7)**

10-1 Rotational Variables

- **Instantaneous angular velocity**: limit as $\Delta t \rightarrow 0$

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} = \frac{d\omega}{dt}. \quad \textbf{Eq. (10-8)}$$

- If the body is rigid, these equations hold for all points on the body
- With right-hand rule to determine direction, angular velocity & acceleration can be written as vectors
- If the body rotates around the vector, then the vector points along the axis of rotation
- Angular displacements are *not* vectors, because the order of rotation matters for rotations around different axes



The order of the rotations makes a big difference in the result.

10-2 Rotation with Constant Angular Acceleration

Learning Objectives

10.14 For constant angular acceleration, apply the relationships between angular position, angular displacement, angular velocity, angular acceleration, and elapsed time (Table 10-1).

10-2 Rotation with Constant Angular Acceleration

- The same equations hold as for *constant linear acceleration*, see Table 10-1
- We simply change linear quantities to angular ones
- Eqs. 10-12 and 10-13 are the basic equations: all others can be derived from them

Table 10-1 Equations of Motion for Constant Linear Acceleration and for Constant Angular Acceleration

Equation Number	Linear Equation	Missing Variable		Angular Equation	Equation Number
(2-11)	$v = v_0 + at$	$x - x_0$	$\theta - \theta_0$	$\omega = \omega_0 + \alpha t$	(10-12)
(2-15)	$x - x_0 = v_0 t + \frac{1}{2}at^2$	v	ω	$\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2$	(10-13)
(2-16)	$v^2 = v_0^2 + 2a(x - x_0)$	t	t	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$	(10-14)
(2-17)	$x - x_0 = \frac{1}{2}(v_0 + v)t$	a	α	$\theta - \theta_0 = \frac{1}{2}(\omega_0 + \omega)t$	(10-15)
(2-18)	$x - x_0 = vt - \frac{1}{2}at^2$	v_0	ω_0	$\theta - \theta_0 = \omega t - \frac{1}{2}\alpha t^2$	(10-16)

Table 10-1

10-2 Rotation with Constant Angular Acceleration

Checkpoint 2

In four situations, a rotating body has angular position $\theta(t)$ given by (a) $\theta = 3t - 4$, (b) $\theta = -5t^3 + 4t^2 + 6$, (c) $\theta = 2/t^2 - 4/t$, and (d) $\theta = 5t^2 - 3$. To which situations do the angular equations of Table 10-1 apply?

Answer: Situations (a) and (d); the others do not have constant angular acceleration

10-3 Relating the Linear and Angular Variables

Learning Objectives

10.15 For a rigid body rotating about a fixed axis, relate the angular variables of the body (angular position, angular velocity, and angular acceleration) and the linear variables of a particle on the body (position, velocity, and acceleration) at any given radius.

10.16 Distinguish between tangential acceleration and radial acceleration, and draw a vector for each in a sketch of a particle on a body rotating about an axis, for both an increase in angular speed and a decrease.

10-3 Relating the Linear and Angular Variables

- Linear and angular variables are related by *r*, perpendicular distance from the rotational axis
- Position (note θ *must* be in radians):

$$s = \theta r$$

Eq. (10-17)

- Speed (note ω *must* be in radian measure):

$$v = \omega r$$

Eq. (10-18)

- We can express the period in radian measure:

$$T = \frac{2\pi}{\omega}$$

Eq. (10-20)

10-3 Relating the Linear and Angular Variables

- Tangential acceleration (radians):

$$a_t = \alpha r$$

Eq. (10-22)

- We can write the radial acceleration in terms of angular velocity (radians):

$$a_r = \frac{v^2}{r} = \omega^2 r$$

Eq. (10-23)

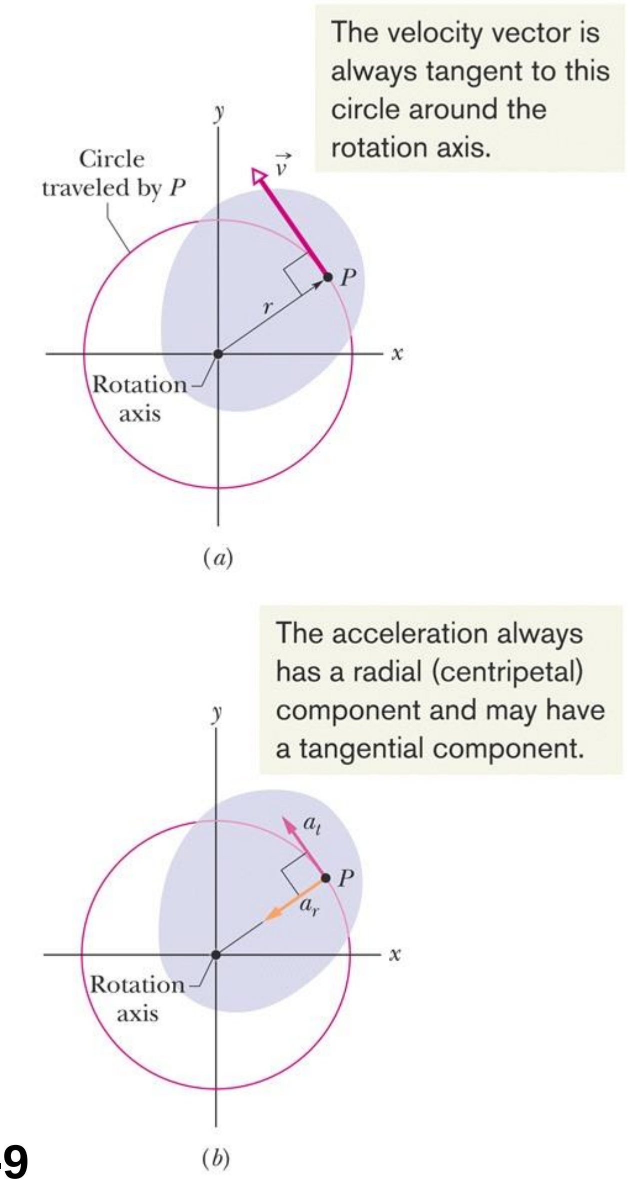


Figure 10-9

10-3 Relating the Linear and Angular Variables

Checkpoint 3

A cockroach rides the rim of a rotating merry-go-round. If the angular speed of this system (*merry-go-round + cockroach*) is constant, does the cockroach have (a) radial acceleration and (b) tangential acceleration? If ω is decreasing, does the cockroach have (c) radial acceleration and (d) tangential acceleration?

Answer: (a) yes (b) no (c) yes (d) yes

10-4 Kinetic Energy of Rotation

Learning Objectives

10.17 Find the rotational inertia of a particle about a point.

10.18 Find the total rotational inertia of many particles moving around the same fixed axis.

10.19 Calculate the rotational kinetic energy of a body in terms of its rotational inertia and its angular speed.

10-4 Kinetic Energy of Rotation

- Apply the kinetic energy formula for a point particle and sum over all the particles $K = \Sigma \, ½ m_i v_i^2$
- Different linear velocities (same angular velocity for all particles but possibly different radii)
- Then write velocity in terms of angular velocity:

$$K = \sum \tfrac{1}{2} m_i (\omega r_i)^2 = \tfrac{1}{2}\left(\sum m_i r_i^2\right)\omega^2, \qquad \textbf{Eq. (10-32)}$$

We call the quantity in parentheses on the right side the **rotational inertia**, or **moment of inertia**, ***I***

- This is a constant for a rigid object and given rotational axis
- Caution: the axis for *I* must always be specified

10-4 Kinetic Energy of Rotation

- We can write:

$$I = \sum m_i r_i^2 \qquad \text{(rotational inertia)}$$

Eq. (10-33)

- And rewrite the kinetic energy as:

$$K = \tfrac{1}{2} I \omega^2 \qquad \text{(radian measure)}$$

Eq. (10-34)

- Use these equations for a finite set of rotating particles
- Rotational inertia corresponds to how difficult it is to change the state of rotation (speed up, slow down or change the axis of rotation)

10-4 Kinetic Energy of Rotation

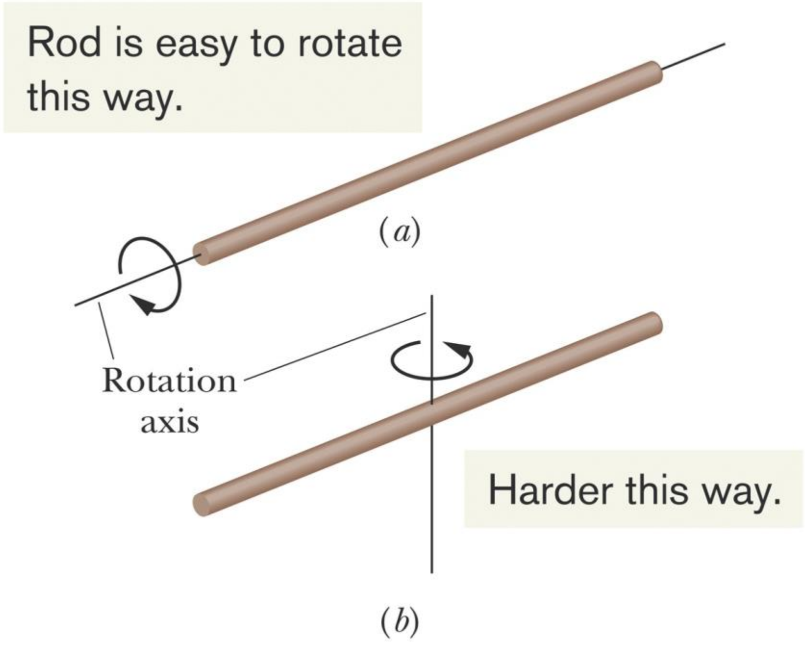
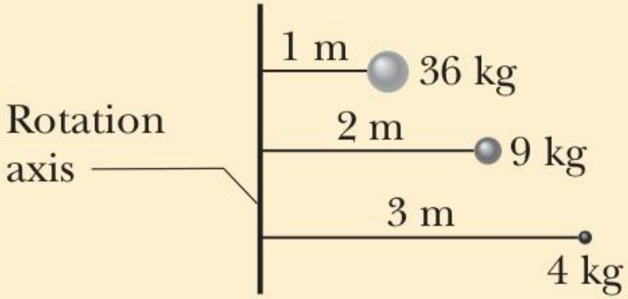


Figure 10-11

Checkpoint 4

The figure shows three small spheres that rotate about a vertical axis. The perpendicular distance between the axis and the center of each sphere is given. Rank the three spheres according to their rotational inertia about that axis, greatest first.



Answer: They are all equal!

10-5 Calculating the Rotational Inertia

Learning Objectives

10.20 Determine the rotational inertia of a body if it is given in Table 10-2.

10.21 Calculate the rotational inertia of body by integration over the mass elements of the body.

10.22 Apply the parallel-axis theorem for a rotation axis that is displaced from a parallel axis through the center of mass of a body.

10-5 Calculating the Rotational Inertia

- Integrating Eq. 10-33 over a continuous body:

$$I = \int r^2 \, dm \quad \text{(rotational inertia, continuous body).}$$ **Eq. (10-35)**

- In principle we can always use this equation
- But there is a set of common shapes for which values have already been calculated (Table 10-2) for common axes

10-5 Calculating the Rotational Inertia

Table 10-2 Some Rotational Inertias

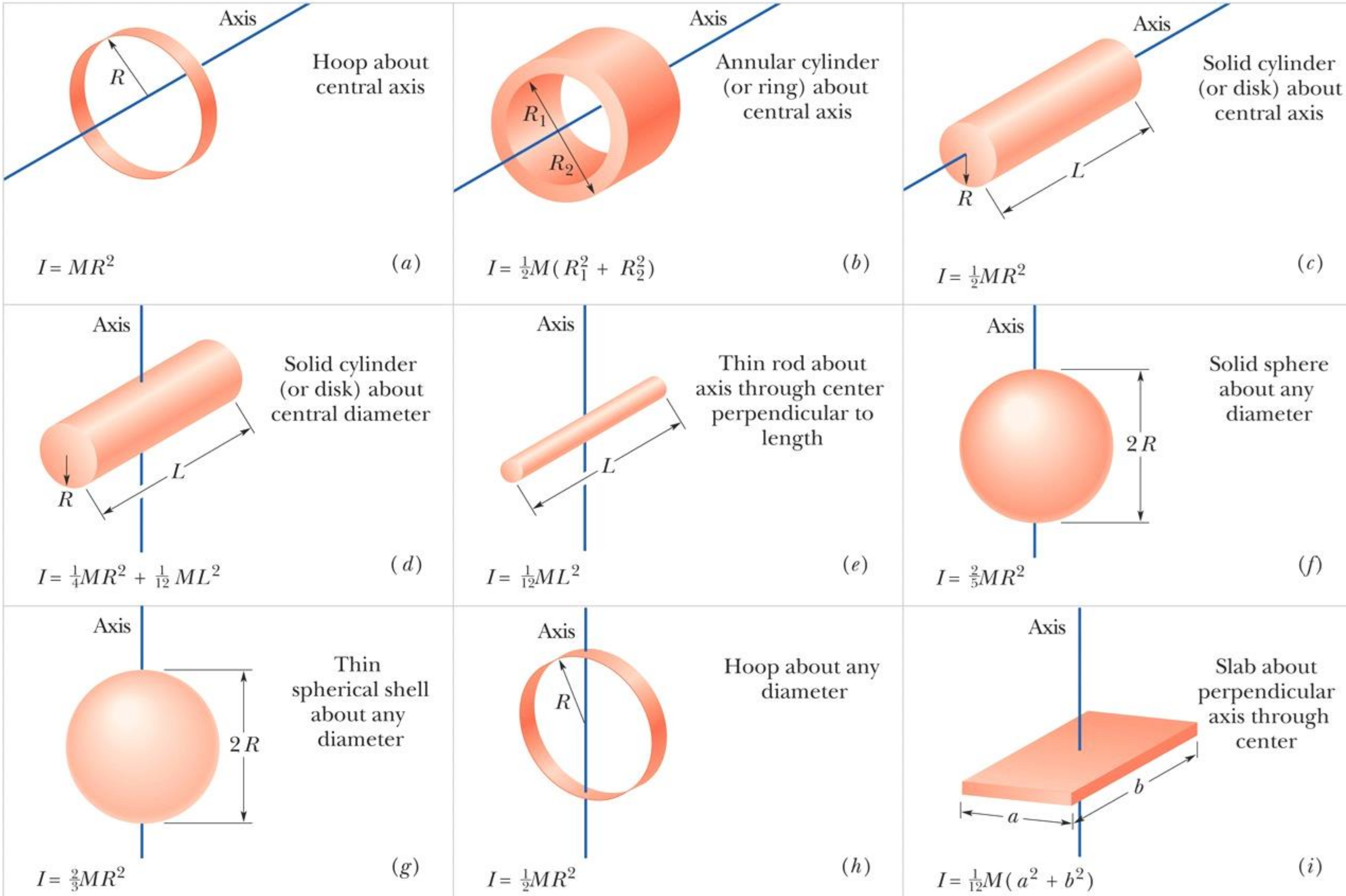


Table 10-2

10-5 Calculating the Rotational Inertia

- If we know the moment of inertia for the center of mass axis, we can find the moment of inertia for a parallel axis with the **parallel-axis theorem**:

$$I = I_{\text{com}} + Mh^2$$ **Eq. (10-36)**

We need to relate the rotational inertia around the axis at P to that around the axis at the com.

- Note the axes *must* be parallel, and the first *must* go through the center of mass
- This does *not* relate the moment of inertia for two arbitrary axes

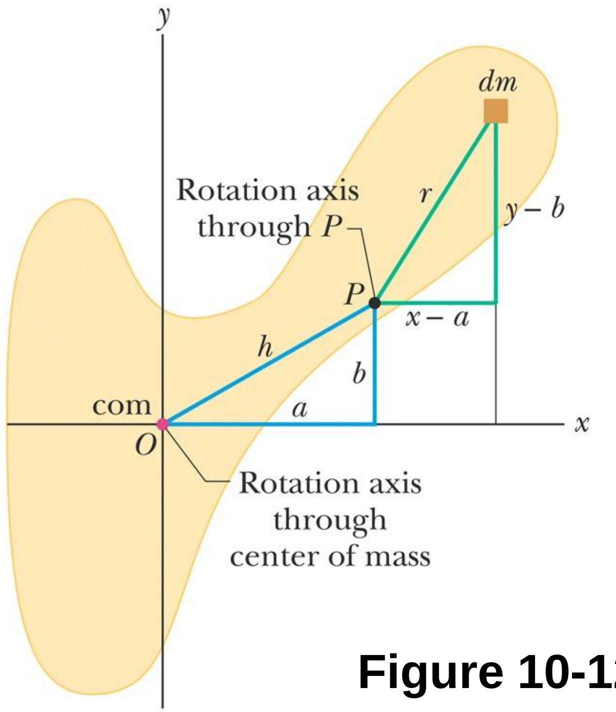
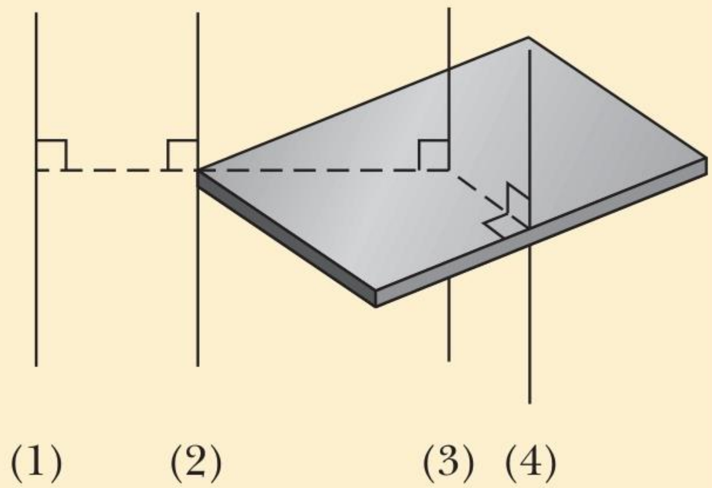


Figure 10-12

10-5 Calculating the Rotational Inertia

Checkpoint 5

The figure shows a book-like object (one side is longer than the other) and four choices of rotation axes, all perpendicular to the face of the object. Rank the choices according to the rotational inertia of the object about the axis, greatest first.



Answer: (1), (2), (4), (3)

Figure 10-13*a* shows a rigid body consisting of two particles of mass m connected by a rod of length L and negligible mass.

(a) What is the rotational inertia I_{com} about an axis through the center of mass, perpendicular to the rod as shown?

Calculations: For the two particles, each at perpendicular distance $\frac{1}{2}L$ from the rotation axis, we have

$$I = \sum m_i r_i^2 = (m)(\tfrac{1}{2}L)^2 + (m)(\tfrac{1}{2}L)^2$$
$$= \tfrac{1}{2}mL^2. \qquad \text{(Answer)}$$

(b) What is the rotational inertia I of the body about an axis through the left end of the rod and parallel to the first axis (Fig. 10-13*b*)?

This situation is simple enough that we can find I using either of two techniques. The first is similar to the one used in part (a). The other, more powerful one is to apply the parallel-axis theorem.

First technique: We calculate I as in part (a), except here the perpendicular distance r_i is zero for the particle on the

10-5 Calculating the Rotational Inertial

Example Calculate the moment of inertia for Fig. 10-13 (b)

- Summing by particle:

$$I = m(0)^2 + mL^2 = mL^2.$$

- Use the parallel-axis theorem

$$I = I_{\text{com}} + Mh^2 = \tfrac{1}{2}mL^2 + (2m)(\tfrac{1}{2}L)^2$$
$$= mL^2.$$

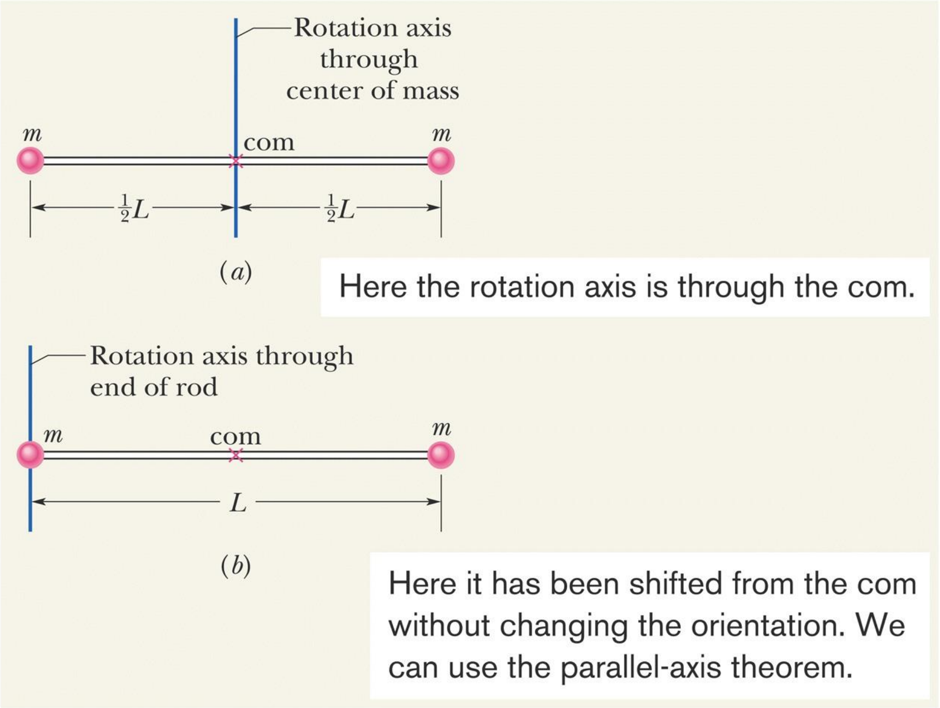


Figure 10-13

10-6 Torque

Learning Objectives

10.23 Identify that a torque on a body involves a force and a position vector, which extends from a rotation axis to the point where the force is applied.

10.24 Calculate the torque by using (a) the angle between the position vector and the force vector, (b) the line of action and the moment arm of the force, and (c) the force component perpendicular to the position vector.

10.25 Identify that a rotation axis must always be specified to calculate a torque.

10.26 Identify that a torque is assigned a positive or negative sign depending on the direction it tends to make the body rotate about a specified rotation axis: "clocks are negative."

10.27 When more than one torque acts on a body about a rotation axis, calculate the net torque.

10-6 Torque

- The force necessary to rotate an object depends on the angle of the force and where it is applied
- We can resolve the force into components to see how it affects rotation

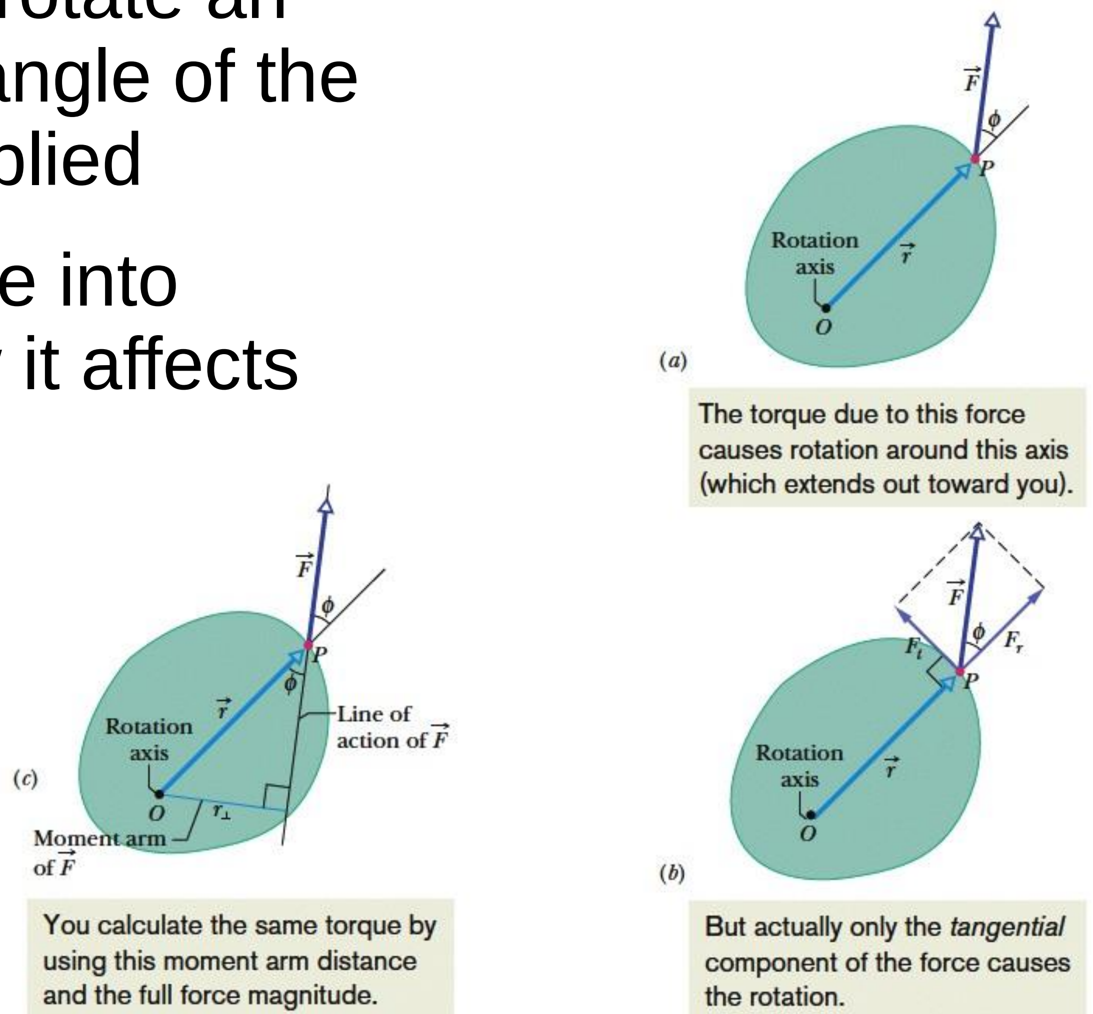


Figure 10-16

10-6 Torque

- **Torque** takes these factors into account:

$$\tau = (r)(F \sin \phi). \qquad \text{Eq. (10-39)}$$

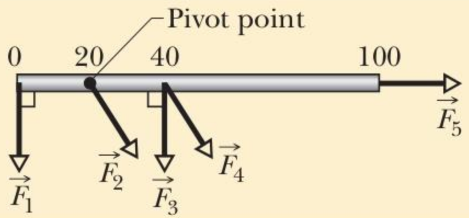
- A line extended through the applied force is called the **line of action** of the force
- The perpendicular distance from the line of action to the axis is called the **moment arm**
- The unit of torque is the newton-meter, N m
- Note that 1 J = 1 N m, but torques are never expressed in joules, torque is not energy

10-6 Torque

- Again, torque is positive if it would cause a counterclockwise rotation, otherwise negative
- For several torques, the **net torque** or **resultant torque** is the sum of individual torques

Checkpoint 6

The figure shows an overhead view of a meter stick that can pivot about the dot at the position marked 20 (for 20 cm). All five forces on the stick are horizontal and have the same magnitude. Rank the forces according to the magnitude of the torque they produce, greatest first.



Answer: F_1 & F_3, F_4, F_2 & F_5

10-7 Newton's Second Law for Rotation

Learning Objectives

10.28 Apply Newton's second law for rotation to relate the net torque on a body to the body's rotational inertia and rotational acceleration, all calculated relative to a specified rotation axis.

10-7 Newton's Second Law for Rotation

- Rewrite *F* = *ma* with rotational variables:

$$\tau_{\text{net}} = I\alpha$$

Eq. (10-42)

- It is torque that causes angular acceleration

The torque due to the tangential component of the force causes an angular acceleration around the rotation axis.

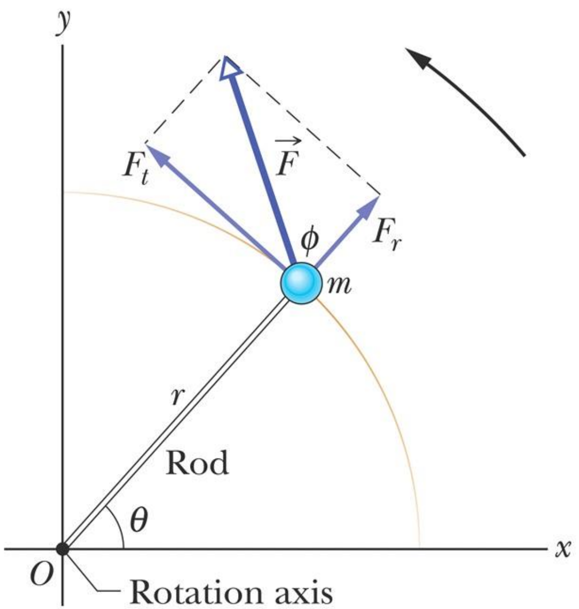
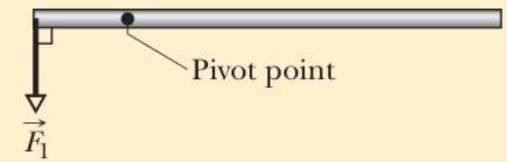


Figure 10-17

10-7 Newton's Second Law for Rotation

Checkpoint 7

The figure shows an overhead view of a meter stick that can pivot about the point indicated, which is to the left of the stick's midpoint. Two horizontal forces, $\vec{F}_1$ and $\vec{F}_2$, are applied to the stick. Only $\vec{F}_1$ is shown. Force $\vec{F}_2$ is perpendicular to the stick and is applied at the right end. If the stick is not to turn, (a) what should be the direction of $\vec{F}_2$, and (b) should F_2 be greater than, less than, or equal to F_1?



Answer: (a) F_2 should point downward, and

(b) should have a smaller magnitude than F_1

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10-8 Work and Rotational Kinetic Energy

Learning Objectives

10.29 Calculate the work done by a torque acting on a rotating body by integrating the torque with respect to the angle of rotation.

10.30 Apply the work-kinetic energy theorem to relate the work done by a torque to the resulting change in the rotational kinetic energy of the body.

10.31 Calculate the work done by a *constant* torque by relating the work to the angle through which the body rotates.

10.32 Calculate the power of a torque by finding the rate at which work is done.

10.33 Calculate the power of a torque at any given instant by relating it to the torque and the angular velocity at that instant.

10-8 Work and Rotational Kinetic Energy

- The rotational work-kinetic energy theorem states:

$$\Delta K = K_f - K_i = \tfrac{1}{2}I\omega_f^2 - \tfrac{1}{2}I\omega_i^2 = W \qquad \textbf{Eq. (10-52)}$$

- The work done in a rotation about a fixed axis can be calculated by:

$$W = \int_{\theta_i}^{\theta_f} \tau \, d\theta \qquad \textbf{Eq. (10-53)}$$

- Which, for a constant torque, reduces to:

$$W = \tau(\theta_f - \theta_i) \qquad \textbf{Eq. (10-54)}$$

10-8 Work and Rotational Kinetic Energy

- We can relate work to power with the equation:

$$P = \frac{dW}{dt} = \tau\omega$$

Eq. (10-55)

- Table 10-3 shows corresponding quantities for linear and rotational motion:

Table 10-3 **Some Corresponding Relations for Translational and Rotational Motion**

Pure Translation (Fixed Direction)		Pure Rotation (Fixed Axis)	
Position	x	Angular position	θ
Velocity	$v = dx/dt$	Angular velocity	$\omega = d\theta/dt$
Acceleration	$a = dv/dt$	Angular acceleration	$\alpha = d\omega/dt$
Mass	m	Rotational inertia	I
Newton's second law	$F_{\text{net}} = ma$	Newton's second law	$\tau_{\text{net}} = I\alpha$
Work	$W = \int F\,dx$	Work	$W = \int \tau\,d\theta$
Kinetic energy	$K = \frac{1}{2}mv^2$	Kinetic energy	$K = \frac{1}{2}I\omega^2$
Power (constant force)	$P = Fv$	Power (constant torque)	$P = \tau\omega$
Work–kinetic energy theorem	$W = \Delta K$	Work–kinetic energy theorem	$W = \Delta K$

Tab. 10-3

10 Summary

Angular Position

- Measured around a **rotation axis**, relative to a **reference line**:

$$\theta = \frac{s}{r}$$ Eq. (10-1)

Angular Displacement

- A change in angular position

$$\Delta\theta = \theta_2 - \theta_1.$$ Eq. (10-4)

Angular Velocity and Speed

- Average and instantaneous values:

$$\omega_{\text{avg}} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t},$$ Eq. (10-5)

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}.$$ Eq. (10-6)

Angular Acceleration

- Average and instantaneous values:

$$\alpha_{\text{avg}} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t},$$ Eq. (10-7)

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}.$$ Eq. (10-8)

10 Summary

Kinematic Equations

- Given in Table 10-1 for constant acceleration
- Match the linear case

Linear and Angular Variables Related

- Linear and angular displacement, velocity, and acceleration are related by *r*

Rotational Kinetic Energy and Rotational Inertia

$$K = \tfrac{1}{2} I\omega^2 \qquad \text{(radian measure)}$$

Eq. (10-34)

$$I = \sum m_i r_i^2 \qquad \text{(rotational inertia)}$$

Eq. (10-33)

The Parallel-Axis Theorem

- Relate moment of inertia around any parallel axis to value around com axis

$$I = I_{\text{com}} + Mh^2$$

Eq. (10-36)

10 Summary

Torque

- Force applied at distance from an axis:

$$\tau = (r)(F \sin \phi).$$ **Eq. (10-39)**

- Moment arm: perpendicular distance to the rotation axis

Work and Rotational Kinetic Energy

$$W = \int_{\theta_i}^{\theta_f} \tau \, d\theta$$ **Eq. (10-53)**

$$P = \frac{dW}{dt} = \tau\omega$$ **Eq. (10-55)**

Newton's Second Law in Angular Form

$$\tau_{\text{net}} = I\alpha$$ **Eq. (10-42)**